BY THE SAME AUTHOR,

RUDIMENTS OF THE LATIN LANGUAGE, 2s. 6d., BOUND.
RUDIMENTS OF THE GREEK LANGUAGE, 4s. BOUND.
London: Whittaker, 1838.

Opinions of the Public Press.

The Student who desires to derive the greatest possible benefit from either of these books, must possess and ponder both, and they will reciprocally explain and teach one another; our reasons for speaking thus will be best conveyed in Mr. Foster's own words: "upon commencing," says he, "the Greek Grammar, it is desirable that the Student who has made some progress in Latin, should find that the knowledge he has acquired is of *direct* assistance to him. To effect this, the author has endeavoured to make the Grammars of the Greek and Latin Languages as similar to each other as possible—by observing the same arrangement throughout—by giving, where it could be done, parallel examples in the nouns, verbs, &c. and by expressing the rules of the Syntax in the same words."

In regard to languages which bear so close an affinity to one another, there are obvious advantages connected with this method, not merely lessening the labour of the teacher, but what is far more important, giving early and interesting lessons in comparative and universal Grammar.—*London Monthly Review.*

The publication of any work tending to simplify the grammars of the dead languages, cannot but be entitled to a favourable reception. It is from this consideration that we welcome the appearance of Mr. Foster's Latin and Greek Grammars. Their structure is singularly simple, and as they are parallel throughout, the pupil who uses them together will find his labours wonderfully abridged, while his mind will be constantly directed to the principles which the two languages have in common.—*London Miscellany.*

These Grammars are superior in their arrangement to those commonly used in schools.—*Athenæum.*

Mr. Foster's Grammars prove him to be a thoughtful, intelligent, and laborious man. His object has been to exhibit the two languages as parallel to each other, a very useful and interesting attempt, which if rightly followed out, cannot fail to make both far more intelligible to the pupil.

We think Mr. F. has occasionally fallen into the temptation of warping the principles of one or the other language to make the parallel more complete, but any errors of this kind do not diminish the worth of his idea, nor make his practical effort to realize it less worthy of attention.

These are rather hints for Mr. F's consideration than objections to his books, which we hope may obtain the attention worthy of the time and reflection he has evidently bestowed upon them.—*Educational Magazine.*

ELEMENTS

OF

ALGEBRA,

FOR THE USE OF

ST. PAUL'S SCHOOL,

SOUTHSEA,

AND ADAPTED TO THE GENERAL OBJECTS OF EDUCATION.

BY WILLIAM FOSTER, M. A.,

Head Master of St. Paul's School, Southsea.

LONDON:

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AND W. WOODWARD, PORTSEA.

1840.

PRINTED BY W. WOODWARD, PORTSEA.

PREFACE.

The design of this little work is to present to the *young* Student the principles of Algebra in the most compendious and simple form. The Author has accordingly introduced nothing but what is absolutely necessary, and endeavoured to state the Rules and the Proofs of them in the plainest manner.

The arrangement of Equations will be found different from that usually adopted: but, as they are placed (see Pp. 15, 26, 39) as soon as the Student has learnt the rules necessary for their solution, the change will be found advantageous in leading the Student to an early application of his knowledge.

A copious collection of Examples will shortly be reprinted, and combined with this work, will, it is hoped, enable the Student to acquire a thorough acquaintance with the Theory and Practice of Algebra.

Southsea, Hants,
July 1, 1840.

DEFINITIONS.

In Algebra the magnitudes of quantities are denoted by *letters*, and their relations by *signs*.

A letter from the *beginning* of the alphabet as a, b, c, &c. denotes a quantity whose value is *known*: from the *end* as x, y, z, denotes one whose value is *unknown*.

The signs are as follows:

$+$, *plus*, signifies *addition*: thus $a+b$ means b added to a.

$-$, *minus*, signifies *subtraction*: thus $a-b$ means b subtracted from a.

$\times$, *into*, signifies *multiplication*: thus $a\times b$ means a multiplied by b.

Sometimes a point is used instead of $\times$: thus $a.b.c$ for $a\times b\times c$. Often and especially with single letters $\times$ is omitted: thus abc for $a\times b\times c$.

$\div$, *by*, signifies *division*: thus $a\div b$ means a divided by b. More usually the dividend is placed over the divisor with a line between them: thus $\frac{a}{b}$ means a divided by b.

$\surd$, *root of*, signifies the *extraction of a root*: a figure over the $\surd$ implies the particular root, and when no figure is expressed 2 is understood: thus $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$ mean the *second, third, fourth root* of a.

( ) a *bracket*, signifies that the quantities it includes are to be considered as *one* term: thus $-(a+b)$ means that the whole quantity $a+b$ is to be subtracted.

A line drawn over, means the same: thus $-\overline{a+b}$ means the same as $-(a+b)$.

$=$ *equal to*, signifies *equality*: thus $x=a-b$ means that x is *equal to* $a-b$.

There are other signs: as $>$ *greater than*: $<$ *less than*: $\because$ *because*: $\therefore$ *therefore*.

TERMS.

A *coefficient* is the number prefixed to a quantity and expresses the number of times it is taken: thus in $2a$, $7xy$, which imply *twice* a and *seven* times xy, the coefficients are 2, 7.

When no coefficient is expressed, 1 is to be understood: thus a, xy mean $1a$, $1xy$.

A *letter* whose value is *known* is often the coefficient of one whose value is *unknown*: thus of ax and by, a and b are the coefficients.

The *power* of a quantity denotes the number of times it is multiplied by itself: thus the 2nd, 3rd, 4th power of a mean a multiplied by a, *once*, *twice*, *three times*.

The *index*, is a small figure placed over the right hand of a quantity to denote the power: thus

a^2 (which stands for $a\times a$) denotes the 2nd power of a
a^3 (............ $a\times a\times a$).......... 3rd ....
a^7 (...... $a\times a\times a\times a\times a\times a\times a$) .... 7th ....

If the *index* be *fractional*, the *denominator* denotes the *root* taken, and the *numerator* the *power* to which the quantity is raised: thus

$a^{\frac{1}{2}}$ denotes the 2nd root of the 1st power of a.
$a^{\frac{3}{4}}$.......... 4th.......... 3rd........ a.

Hence $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{5}{7}}$ mean the same as $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[7]{a^5}$.

Positive quantities are those whose signs are $+$: as $+2a$, $+10ax$.

Negative quantities are those whose signs are $-$: as $-2a$, $-10ax$.

When no sign is expressed, $+$ is understood: thus $2a$ means $+2a$.

$+$ is omitted only when a quantity stands *alone*, or at the *beginning* of an expression: thus we write $3a$ not $+3a$: $a+b-c$ not $+a+b-c$.

Quantities have *like signs*, when the signs are all $+$, or all $-$: as $7a$, $6b$: or $-6x, -y, -10a$.

Quantities have *unlike signs*, when some of the signs are $+$, and some $-$: as a, $-3b$, $7c$.

Like quantities are those which have the *same letters* with the *same indices*: as $5a$, $7a$: and a^2x, $16a^2x$, $-4a^2x$.

Unlike quantities are those which have *different letters*: as $3a$, $2b$: or the *same letters* with *different indices*: as $3ax^2$, $2a^2x^3$.

A *Term* is one of any quantities connected by $+$ or $-$: thus in $a+2b-3c$ the terms are a, $2b$, $-3c$.

A *simple* quantity consists of *one* term: as $2a$.
A *binomial*............ 2 terms: as $2a+b$.
A *trinomial* .... 3 terms: as $2a+b-c$.
A *quadrinomial*........ 4 terms: as $2a+b-c+d$.
A *multinomial* of more than 4 terms.

NOTATION.

Notation is the finding the Numerical value of an Algebraical expression.

RULE. For the letters substitute their given values, and reduce the expression to its simplest form.

Ex. 1. If $a=6$. $b=5$. $c=4$.

Find the value of $\frac{a^3b}{a+3c}+c^3$.

$$\frac{a^3b}{a+3c}+c^3=\frac{6^3\times5}{6+3\times4}+4^3=\frac{216\times5}{6+12}+64=\frac{1080}{18}+64$$
$$=60+64=124.$$

Ex. 2. Find the value of $a^2\times(a+b)-2abc$.

$a^2\times(a+b)-2abc=6^2\times(6+5)-2\times6\times5\times4=36\times11$ $-240=396-240=156.$

Ex. 3. Find the value of $\sqrt{(2a^2-\sqrt{2ac+c^2})}$.

$\sqrt{(2a^2-\sqrt{2ac+c^2})}=\sqrt{(2\times6^2-\sqrt{2\times6\times4+4^2})}=\sqrt{(2\times36-\sqrt{48+16})}=\sqrt{(72-\sqrt{64})}=\sqrt{(72-8)}=\sqrt{64}=8.$

ADDITION.

Addition is divided into three cases.

CASE I.

When the quantities are like with like signs.

RULE. Add the coefficients together, to their sum annex the literal part, and prefix the common sign.

Ex. 1.
$$\begin{array}{r} 5a \\ 4a \\ \hline 9a \\ \hline \end{array}$$

Ex. 2.
$$\begin{array}{r} -b \\ -3b \\ -2b \\ \hline -6b \\ \hline \end{array}$$

Ex. 3.
$$\begin{array}{r} 2a+x^2 \\ 3a+x^2 \\ \frac{1}{2}a+\frac{2}{3}x^2 \\ \hline \frac{11}{2}a+\frac{8}{3}x^2 \\ \hline \end{array}$$

PROOF. *Ex.* 1. $5a$ to be added and $4a$ to be added must together be $9a$ to be added, and therefore the sum is $9a$. *Ex.* 2. $1b$, to be subtracted, $3b$ to be subtracted, and $2b$ to be subtracted must together be $6b$ to be subtracted, and therefore their sum is $-6b$.

CASE II.

When the quantities are like with unlike signs.

RULE. Add the positive coefficients, and then the negative ones: subtract the sums thus found: to the difference annex the literal part, and prefix the sign of the greater sum.

Ex. 1.
$$\begin{array}{r} -4ay+z^2 \\ -3ay+3z^2 \\ 11ay-2z^2 \\ 13ay-2z^2 \\ \hline 17ay \quad * \\ \hline \end{array}$$

Ex. 2.
$$\begin{array}{r} \frac{1}{2}a^2+\frac{1}{2}(a+b) \\ -a^2-3(a+b) \\ a^2-\frac{1}{2}(a+b) \\ \frac{3}{2}a^2+6(a+b) \\ \hline 2a^2\ +(a+b) \\ \hline \end{array}$$

PROOF. *Ex.* 1. $11ay$ and $13ay$ to be added are together $24ay$ to be added: $4ay$ and $3ay$ to be subtracted are together $7ay$ to be subtracted: the whole therefore is $24ay$ to be added and $7ay$ to be subtracted, *i. e.*, $17ay$ to be added: z^2 and $3z^2$ to be added are together $4z^2$ to be added: $2z^2$ and $2z^2$ to be subtracted are $4z^2$ to be subtracted: the whole therefore is $4z^2$ to be added and $4z^2$ to be subtracted, *i. e.* 0.

Case III.

When the quantities and signs are some like and some unlike.

Rule. Collect (by Case 1 and 2) the like quantities, if any: and to their sum annex the unlike quantities each with its proper sign.

Ex. 1.

$$\begin{array}{lll} -2a & & \\ & +b & \\ 5a & & -4d \\ \hline 3a & +b & -4d \end{array}$$

Ex. 2.

$$\begin{array}{l} \frac{3}{4}(c^2+d^2) \\ -(c^2+d^2)+x^2 \\ 2(c^2+d^2)-x^2+x \\ \hline \frac{7}{4}(c^2+d^2) \ * \ +x \end{array}$$

Proof. The sum of $-2a$ and $5a$ is (by Case 2) $3a$: also b is to added, and $4d$ subtracted: the whole therefore is $3a+b-4d$.

N. B.—Quantities with literal coefficients, are added by putting the coefficients in a bracket, and annexing the common letter.

Ex.

$$\begin{array}{r} ax+cz^2 \\ -bx+ez^2 \\ -fz^2 \\ \hline (a-b)x+(c+e-f)z^2. \end{array}$$

SUBTRACTION.

Rule. Change, or conceive to be changed, all the signs of the quantity to be subtracted, and then proceed as in addition.

Ex. 1. $\begin{array}{r} 6a+b \\ 2a-3b \\ \hline 4a+4b \end{array}$ *Ex.* 2. $\begin{array}{r} a-\frac{3}{4}c \\ b+\frac{1}{4}c \\ \hline a-b-c \end{array}$ *Ex.* 3. $\begin{array}{r} 3(a-b)+c \\ 2a+3b-5c \\ \hline a-6b+6c \end{array}$

Proof. *Ex.* 1. $2a$ taken from $6a+b$, will leave $4a+b$: but we had to subtract not $2a$, but a quantity less than $2a$ by $3b$: therefore the result $4a+b$ must be too little by $3b$, *i. e.* it must be $4a+4b$. Now we should obtain the same result by changing $2a-3b$ into $-2a+3b$, and adding it to the upper line.

N. B.—When a bracket is removed after a *negative* sign, the signs of all the quantities under it must be changed: thus $6a+b-(2a-3b)=6a+b-2a+3b$.

MULTIPLICATION.

Multiplication is divided into three cases.

CASE I.

When both the Multiplier and Multiplicand are simple quantities.

RULE. Attend to three things, the signs, the coefficients, the letters.

1mo. Like signs produce $+$, and unlike, $-$.

2do. Multiply the coefficients as in whole numbers.

3tio. If the letters be the *same,* add their indices: if *different,* write the letters down one after the other.

Ex. 1. Multiply $12a$ by $-3b$:

$$12a\times-3b=-36ab.$$

Ex. 2. Multiply $-4a^2x^3y^5$ by $-2a^6x^2y^3$

$$-4a^2x^3y^5\times-2a^6x^2y^3=8a^{2+6}x^{3+2}y^{5+3}=8a^8x^5y^8.$$

Ex. 3. Multiply $-5a^{-2}x^{\frac{1}{2}}y^{-5}$ by $-\frac{2}{3}a^3x^{\frac{3}{2}}y^{-7}$.

$$-5a^{-2}x^{\frac{1}{2}}y^{-5}\times-\frac{2}{3}a^3x^{\frac{3}{2}}y^{-7}=\frac{10}{3}a^{-2+3}x^{\frac{1}{2}+\frac{3}{2}}y^{-5-7}=\frac{10}{3}ax^2y^{-12}.$$

PROOF. Rule 1mo. will appear from considering that Multiplication is only another form of Addition.

1. $+2\times+3$ means that $+2$ is to be added 3 times and therefore equals $+6$.

2. $+2\times-3$ means that $+2$ is to be subtracted 3 times and therefore equals -6.

3. $-2\times+3$ means that -2 is to be added 3 times and therefore equals -6.

4. -2×-3 means that -2 is to be subtracted 3 times and therefore equals $+6$.

For the same reason $+a \times +b = +ab$: $+a \times -b = -ab$: $-a \times +b = -ab$: $-a \times -b = +ab$.

Cor. The sign for *any* number of factors may be found by the same rule: thus

$$+a \times +b \times -c = +ab \times -c = -abc.$$
$$+a \times +b \times -c \times -d = +ab \times +cd = +abcd.$$

To prove Rule 2do. we premise that it is immaterial in what order the letters are placed: thus

$$abc = bca \text{ or } acb \text{ or } bac.$$

just as $2 \times 3 \times 4 = 3 \times 4 \times 2$ or $2 \times 4 \times 3$ or $3 \times 2 \times 4$.

Hence $5a \times 3b = 5 \times 3 \times a \times b = 15ab$.

Rule 3tio. appears from the definition of an index in page 4: thus

$a^2 \times a^3 = (a \times a) \times (a + a \times a) = a \times a \times a \times a \times a = a^5$, *i.e.* a^{2+3}.
$a^n \times a^m = (a \times a \times a$ repeated n times$)$ $\times (a \times a \times a$ repeated m times$) = a \times a \times a$ repeated $n + m$ times $= a^{n+m}$.

Case II.

When the Multiplicand is a compound and the Multiplier a simple quantity.

Rule. Multiply each term of the multiplicand by the multiplier, (by Case I.) and set the products down one after another.

Ex. 1. Multiply $a+b$ by c.

$$(a+b) \times c = ac + bc.$$

Ex. 2. Multiply $7a + 3ax - 4b^2x$ by $6abx$.

$$(7a + 3ax - 4b^2x) \times 6abx = 42a^2bx + 18a^2bx^2 - 24ab^3x^2.$$

Ex. 3. Multiply $3a^3b^{-2} + \frac{5}{6}a^4c$ by $\frac{3}{2}a^{-1}b^{-2}c^3$.

$$(3a^3b^{-2} + \tfrac{5}{6}a^4c) \times \tfrac{3}{2}a^{-1}b^{-2}c^3 = \tfrac{9}{2}a^2b^{-4}c^3 + \tfrac{5}{4}a^3b^{-2}c^4.$$

Proof. $a+b$ is to be added c times: *i. e.* a is to be added c times and $+b$ is to be added c times and the result is therefore $ac+bc$.

Case III.

When both the Multiplicand and Multiplier are compound quantities.

Rule. Multiply the multiplicand by each term of the multiplier, (by Case 2.) and collect all the products into one sum.

Ex. 1. Multiply $a+b$ by $c-d$.

$$\begin{array}{l} a+b \\ c-d \\ \hline ac+bc \\ \qquad\quad -ad-bd \\ \hline ac+bc-ad-bd \end{array}$$

Ex. 2. Multiply $5a+3x+y$ *by* $3a+2x$.

$$\begin{array}{l} 5a+3x+y \\ 3a+2x \\ \hline 15a^2+9ax+3ay \\ \qquad +10ax+6x^2+2xy \\ \hline 15a^2+19ax+3ay+6x^2+2xy \end{array}$$

Proof. $a+b$ is to be taken $c-d$ times: *i. e.* it is to be added c times and subtracted d times, and the result is therefore $ac+bc-ad-bd$.

$$\begin{array}{lll} a+b & a-b & a+b \\ a+b & a-b & a-b \\ \hline a^2+ab & a^2-ab & a^2+ab \\ \quad +ab+b^2 & \quad -ab+b^2 & \quad -ab-b^2 \\ \hline a^2+2ab+b^2 & a^2-2ab+b^2 & a^2 \quad * \quad -b^2 \end{array}$$

Hence 1. The square of the sum of two quantities equals the square of the first, plus twice the product of the quantities, plus the square of the second.

2. The square of the difference of two quantities equals the square of the first, minus twice the product of the quantities, plus the square of the second.

3. The product of the sum and difference of two quantities equals the square of the first, minus the square of the second.

DIVISION.

Division is divided into three cases.

Case I.

When both the dividend and divisor are simple quantities.

Rule. Attend to three things, the signs, the coefficients, the letters.

1mo. Like signs produce $+$, and unlike $-$.

2do. Divide the coefficients as in Arithmetic.

3tio. If the letters be the *same*, subtract the index of the divisor from that of the dividend: if *different*, place the letters of the divisor under those of the dividend, in the form of a fraction.

Ex. 1. Divide $15a^5$ by $5a^2$.

$$15a^5 \div 5a^2 = 3a^{5-2} = 3a^3.$$

Ex. 2. Divide $-18a^2x^3y$ by $9ax^2z$.

$$-18a^2x^3y \div 9ax^2z = \frac{-2a^{2-1}x^{3-2}y}{z} = -\frac{2axy}{z}.$$

Ex. 3. Divide $\frac{5}{6}a^{-5}x^3y^m$ by $\frac{3}{2}ax^{-1}y^{-n}$.

$$\tfrac{5}{6}a^{-5}x^3y^m \div \tfrac{3}{2}ax^{-1}y^{-n} = \tfrac{5}{9}a^{-5-1}x^{3+1}y^{m+n} = \tfrac{5}{9}a^{-6}x^4y^{m+n}.$$

Proof. Rule 1, will appear from the rule for the signs in Multiplication.

$$\left.\begin{array}{ll} 1. & \dfrac{+ab}{+a} = \dfrac{+a\times+b}{+a} = +b. \\[2ex] 2. & \dfrac{-ab}{-a} = \dfrac{-a\times+b}{-a} = +b. \end{array}\right\} \text{i.e. like signs, produce } +.$$

$$\left.\begin{array}{ll} 3. & \dfrac{+ab}{-a} = \dfrac{-a\times-b}{-a} = -b. \\[2ex] 4. & \dfrac{-ab}{+a} = \dfrac{+a\times-b}{+a} = -b. \end{array}\right\} \text{i.e. unlike signs, produce} -.$$

Rules 2 and 3 are similarly proved.

$$5a\times 3b=15ab\ \therefore \frac{15ab}{5a}=3b.$$

$$\frac{a^3}{a^2}=\frac{a\times a\times a}{a\times a}=a.\ i.e.\ a^{3-2}.$$

$$\frac{a^m}{a^n}=\frac{a\times a\times a \text{ repeated } m \text{ times}}{a\times a\times a \text{ repeated } n \text{ times}}=a\times a\times a \text{ repeated } m-n \text{ times}=a^{m-n}.$$

Cor. 1. Any quantity to the power of $0=1$:

For $\frac{a^n}{a^n}=1$: but $\frac{a^n}{a^n}=a^{n-n}=a^0$.

$\therefore\quad a^0=1.$

Cor. 2. $\quad a^{-n}=\frac{1}{a^n}.$

For, $\frac{a^m}{a^n}=a^{m-n}$ and $\therefore$ if $m=0$, $\frac{a^0}{a^n}=a^{0-n}=a^{-n}$.

but $\frac{a^0}{a^n}=\frac{1}{a^n}$ (Cor. 1.) $\therefore\ \frac{1}{a^n}=a^{-n}$.

Vice versâ, $a^n=\frac{1}{a^{-n}}$.

Cor. 3. A quantity may be removed from the numerator of a fraction to the denominator, and vice versâ, by changing the sign of its index.

Since $a^2=\frac{1}{a^{-2}}$ and $b^{-3}=\frac{1}{b^3}$ (Cor. 2).

$$\frac{a^2}{b^3}=a^2b^{-3} \text{ or } \frac{1}{a^{-2}b^3} \text{ or } \frac{b^{-3}}{a^{-2}}.$$

CASE II.

When the Dividend is a compound, and the Divisor a simple quantity.

RULE. Divide each term of the dividend by the divisor (by Case 1).

Ex. 1. Divide $ac+bc$ by c.

$$(ac+bc)\div c=a+b.$$

Ex. 2. Divide $3a^3b^4c^4-12ab^3c+15a^5b^8$ by $3ab^2c$.

$$(3a^3b^4c^4-12ab^3c+15a^5b^8)\div 3ab^2c=a^2b^2c^3-4b+\frac{5a^4b^6}{c}.$$

Ex. 3. Divide $\frac{3}{4}a^{-2}x^5-\frac{3}{5}a^m c^2x^{-1}$ by $2a^3x^{-2}$.

$$(\tfrac{3}{4}a^{-2}x^5-\tfrac{3}{5}a^m c^2x^{-1})\div 2a^3x^{-2}=\tfrac{3}{8}a^{-5}x^7-\tfrac{3}{10}a^{m-3}c^2x.$$

PROOF. $(a+b)\times c=ac+bc$, by Case II of Multiplication $\therefore (ac+bc)\div c=a+b$.

CASE III.

When both the Dividend and Divisor are compound quantities.

RULE. Arrange the terms of the divisor and of the dividend according to the powers of some one letter common to both, beginning with its highest power: find how often the first term of the divisor is contained in that of the dividend, and place the result in the quotient: multiply the divisor by this quotient: place the product under the dividend and then subtract it.

Bring down as many terms of the dividend as may be necessary: consider this remainder as a new dividend and proceed as before.

Ex. 1. Divide $3a^2b+a^3+b^3+3ab^2$ by a^2+b^2+2ab.

$$
\begin{array}{l}
a^2+2ab+b^2)a^3+3a^2b+3ab^2+b^3(a+b. \\
\qquad\qquad\quad \underline{a^3+2a^2b+ab^2} \\
\qquad\qquad\qquad\quad a^2b+2ab^2+b^3 \\
\qquad\qquad\qquad\quad \underline{a^2b+2ab^2+b^3} \\
\qquad\qquad\qquad\quad \underline{\;*\quad\;*\quad\;*\;}
\end{array}
$$

We arrange the divisor and dividend according to the powers of a, putting a^2 before $2ab$, and $2ab$ before b^2: as also, a^3 before $3a^2b$, &c. Now a^2 the first term of the divisor is contained a times in a^3 the first term of the dividend, and therefore a is the first term the quotient: we multiply the divisor by a and place the product $a^3+2a^2b+ab^2$ under the dividend and subtract: the remainder is a^2b+2ab^2 to which we bring down $+b^3$. As a^2 is contained $+b$ times in a^2b, the first term of the remainder, we place $+b$ in the quotient, and multiply and subtract as before: there is no remainder and the operation is concluded.

If there be a remainder, it with the divisor underneath it must be affixed to the quotient.

Ex. 2. Divide a^3-x^3 by $a+x$.

$$
\begin{array}{l}
a+x)a^3-x^3(a^2-ax+x^2-\dfrac{2x^3}{a+x}. \\
\qquad\quad \underline{a^3+a^2x} \\
\qquad\quad -a^2x-x^3 \\
\qquad\quad \underline{-a^2x-ax^2} \\
\qquad\qquad\qquad ax^2-x^3 \\
\qquad\qquad\qquad \underline{ax^2+x^3} \\
\qquad\qquad\qquad\quad \underline{-2x^3}
\end{array}
$$

EQUATIONS.

1. An equation is an equality which exists amongst quantities, some of which are known and others unknown, which equality is denoted by the sign $=$: thus

(1) $x=a+b$.

(2) $2x+4=6x+7-9a$.

2. The *terms* of an equation are the quantities of which it is composed: thus in (2) the terms are $2x$, $+4$, $6x$, $+7$, $-9a$.

3. Equations are classed according to the *highest* power of the unknown quantity which they contain: thus in a *simple* equation the *highest* power is 1: as $x=5$.
quadratic.......................2: as $x^2+2x=10$.
cubic ....3: as $x^3-3x^2=4x-5$.

4. The *root* of an equation is that quantity which, when substituted for the unknown quantity, makes both sides of the equation equal: thus in $x+3=7$, the root is 4: for, substituting 4 for x, we have $4+3=7$.

5. The *solution* of an equation is the finding its root, or the value of the unknown quantity.

On the solution of simple equations.

6. In the equation $x=a$, the value of x is known: if therefore we can reduce an equation to this form, the value of the unknown quantity will be known. This may be done by rules founded upon the axioms,

1. If the same be added to equals, the sums will be equal.

2. If the same be taken from equals, the remainders will be equal.

3. If equals be multiplied by the same, the products will be equal.

4. If equals be divided by the same, the quotients will be equal.

Rule I.

7. Any term may be transposed from one side of an equation to the other by changing its sign.

Ex. 1. If $x+3=7$
then $x=7-3$.

Ex. 2. If $4x-8=3x+20$
then $4x-3x=20+8$.

Proof. The same quantity being added to, or subtracted from each side, the results are equal, by *Ax.* 1 and 2. Thus in *Ex.* 1. if $x+3=7$, then subtracting 3 from both sides $x+3-3=7-3$. *i. e.* $x=7-3$. In *Ex.* 2. 8 is added to, and $3x$ subtracted from each side.

Rule II.

8. If each side of an equation be divided by the same quantity, the results will be equal.

Ex. If $5x=10$.
then, dividing by $5, x=2$.

Proof. Equals being divided by the same quantity 5, the quotients are equal, by *Ax.* 4.

N. B.—These two rules are sufficient for the solution of equations of the *simplest* forms.

Ex. 1. $3x-5=23-x$
transposing, $3x+x=23+5$
$\therefore 4x=28$
dividing by 4, $x=7$

Ex. 2. $6x-20=100+4x-3x$
transposing $6x-4x+3x=100+20$
$\therefore 5x=120$
dividing by 5, $x=24$.

On solving Problems by means of Equations.

The solution of questions is effected by expressing *algebraically*, what in the question is expressed by words. To do this, first represent the quantity sought by x: next find expressions for the several quantities mentioned in the question: then express the relation between these quantities by an equation.

Ex. 1. What is the number to the double of which if 18 be added, the sum will be 82?

Let x=the number

then $2x+18$=double the number with 18 added.

Now this is to be equal to 82,

$\therefore 2x+18=82$

whence $2x=82-18=64$

and $x=32$, the number required.

Ex. 2. A, B, C, have together 76£: B has 10£ more than A, and C as much as A and B together. How much had each?

Let $x=$ what A had.

then $x+10=$ what B had.

and $2r+10=$ what C had.

Now they had together 76£.

$\therefore 4x+20=76$

$\therefore 4x=56$

and $x=14$£ what A had.

$x+10=24$£ what B had.

$2x+10=38$£ what C had.

FRACTIONS.

To find the greatest common measure of two quantities.

1. A *common measure* is that quantity which will divide two or more quantities without remainder: the *greatest* quantity which will do so, is the *greatest common measure*: thus of $2abxy$, $4bx^2y$, $6abx^2y^2$ the *common measures* are 2, $2x$, $2bx$, $2bxy$, and the *greatest common measure* is $2bxy$.

2. Of simple quantities, and some compound ones, the greatest common measures may be found at once: thus,
of $20\,a^2d$ and $5ax$ it is $5a$:
of $(a^2-b^2)x^2y$ and $(a^2+2ab+b^2)\,y^2$ it is $(a+b)y$.

3. In other cases, the greatest common measure may thus be found:

Rule. Arrange the quantities according to the powers of some letter: divide the greater by the less, and the preceding divisor by the last remainder, and so on till there is no remainder, the last divisor will be the greatest common measure.

Ex. Find the greatest common measure of $2x+1+x^2$ and $2x^2+x^3+2x+1$.

$$
\begin{array}{l}
x^2+2x+1)x^3+2x^2+2x+1(x \\
\qquad\qquad\quad \underline{x^3+2x^2+x} \\
\qquad\qquad\qquad\qquad x+1)x^2+2x+1(x+1 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{x^2+\ x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\ x+1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\ \underline{x+1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\ \ *\quad *
\end{array}
$$

We arrange the quantities according to the powers of x: we then divide, and have a remainder $x+1$: we divide the former divisor x^2+2x+1 by this remainder: there is no remainder, and $x+1$ is therefore the greatest common measure.

4. If the first term of any divisor is not contained exactly in the first term of the dividend, multiply the dividend by such a quantity, as will make the division succeed: also, a factor not common to both the divisor and dividend may be left out, since it cannot enter into the common measure.

Ex. Find the greatest common measure of $8y^3-4y^2-2y+1$ and $12y^3+4y^2-3y-1$.

$$\begin{array}{r}
12y^3+4y^2-3y-1\\
2\\
\hline
8y^3-4y^2-2y+1)\,24y^3+8y^2-6y-2\,(3\\
24y^3-12y^2-6y+3\\
\hline
20y^2 \qquad -5
\end{array}$$

dividing by 5, $$\begin{array}{r}
4y^2-1)\,8y^3-4y^2-2y+1\,(2y-1\\
8y^3 \qquad -2y \qquad\quad\\
\hline
-4y^2 \qquad +1\\
-4y^2 \qquad +1\\
\hline
* \qquad\quad *
\end{array}$$

$\therefore$ $4y^2-1$ is the greatest common measure.

As $8y^3$ is not exactly contained in $12y^3$, we multiply by 2. The remainder is $20y^2-5$, which we divide by 5, a quantity which will divide it, but not the dividend.

Proof. See Appendix.

REDUCTION OF FRACTIONS.

LEMMA 1. A fraction is multiplied by a whole number, by multiplying the numerator by it: thus $\frac{2}{3}\times 5=\frac{10}{3}$: for the product must be 5 times as great as $\frac{2}{3}$, which $\frac{10}{3}$ is, since 10 is 5 times as great as 2, whilst the divisor 3 is unchanged: similarly $\frac{a}{b}\times c=\frac{ac}{b}$.

2. A fraction is divided by a whole number, by multiplying the denominator by it: thus $\frac{2}{3}\div 5=\frac{2}{15}$: for the quotient must be 5 times as small as $\frac{2}{3}$, which $\frac{2}{15}$ is, since 2 remains the same, whilst the divisor is 5 times as great as before: similarly $\frac{a}{b}\div c=\frac{a}{bc}$.

3. Similarly, a fraction may be multiplied by a whole number by dividing the denominator, and divided by dividing the numerator: thus $\frac{4}{6}\times 3=\frac{4}{2}$: and $\frac{6}{4}\div 3=\frac{2}{4}$.

4. If the numerator and denominator of a fraction be both multiplied or divided by the same quantity, the value of the fraction is not altered: thus $\frac{a}{b}=\frac{ac}{bc}$: for (*Lem.* 1) when we multiply the numerator by c, we multiply the fraction by c, and (*Lem.* 2) when we multiply the denominator by c, we divide the fraction by c: therefore, as we multiply and divide the fraction by the same quantity c, we do not alter the value of the fraction. Similarly $\frac{a}{b}=\frac{\frac{a}{c}}{\frac{b}{c}}$.

Case I.

To reduce a fraction to its lowest terms.

Rule. Divide the numerator and denominator by their greatest common measure.

Ex. 1. Reduce $\dfrac{ax^2y}{4a^2cxy}$ to its lowest terms.

The greatest common measure is axy:

$$\therefore \frac{ax^2y}{4a^2cxy} = \frac{x}{4ac}.$$

Ex. 2. Reduce $\dfrac{x^3-b^2x}{x^2+2bx+b^2}$ to its lowest terms.

$$\begin{array}{l} \quad x^3-b^2x \\ \text{dividing by } x,\ x^2-b^2)\,x^2+2bx+b^2\,(1 \\ \qquad\qquad\qquad\quad \underline{x^2 \qquad\quad -b^2} \\ \qquad\qquad\qquad\qquad 2bx+2b^2 \\ \qquad\quad \text{dividing by } 2b,\ x+b)\,x^2-b^2\,(x-b \\ \qquad\qquad\qquad\qquad\qquad\quad \underline{x^2+bx} \\ \qquad\qquad\qquad\qquad\qquad\quad -bx-b^2 \\ \qquad\qquad\qquad\qquad\qquad\quad \underline{-bx-b^2} \\ \qquad\qquad\qquad\qquad\qquad\qquad * \quad * \end{array}$$

$\therefore$ The greatest common measure is $x+b$,

$$\text{and } \frac{2bx+2b^2}{x^2-b^2} = \frac{2b}{x-b}.$$

Proof. *Lem.* 4

Case II.

To reduce a mixed number to an improper fraction.

Rule. Multiply the integral part by the denominator of the fractional part: to the product annex the numerator of the fractional part (adding or subtracting as the sign before the fraction is + or −), and under the result place the denominator.

Ex 1. Reduce $a+\frac{b}{c}$ to an improper fraction.

$$a+\frac{b}{c}=\frac{ac+b}{c}.$$

Ex. 2. Reduce $a-b-\frac{2ab+b^2}{a-b}$ to an improper fraction.

$$a-b-\frac{2ab+b^2}{a-b}=\frac{(a-b)\times(a-b)-2ab-b^2}{a-b}$$

$$=\frac{a^2-2ab+b^2-2ab-b^2}{a-b}=\frac{a^2-4ab}{a-b}.$$

Proof. a or $\frac{a}{1}=\frac{ac}{c}$ (*Lem.* 4) $\therefore a+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c}$.

Case III.

To reduce an improper fraction to a mixed number.

Rule. Divide the numerator by the denominator: the quotient will be the integral part, and the remainder with the denominator under it the fractional part.

Ex. 1. Reduce $\frac{ac+d}{c}$ to a mixed number.

$$\frac{ac+d}{c}=a+\frac{d}{c}.$$

Ex. 2. Reduce $\frac{10x^2-5x-a+3}{5x}$ to a mixed number.

$$\frac{10x^2-5x-a+3}{5x}=2x-1-\frac{a-3}{5x}.$$

Proof. Since $a+\frac{d}{c}=\frac{ac+d}{c}$ (Case II) $\therefore \frac{ac+d}{c}=a+\frac{d}{c}$.

Case IV.

To reduce fractions to others of equal value having a common denominator.

Rule. Multiply the numerator and denominator of each fraction, by all the other denominators.

Ex. Reduce $\frac{a}{2c}$, $\frac{3b}{d}$, $\frac{4b^2}{3}$ to a common denominator.

$$\frac{a}{2c}=\frac{a}{2c}\times\frac{d}{d}\times\frac{3}{3}=\frac{3ad}{6cd}.$$

$$\frac{3b}{d}=\frac{3b}{d}\times\frac{2c}{2c}\times\frac{3}{3}=\frac{18bc}{6cd}.$$

$$\frac{4b^2}{3}=\frac{4b^2}{3}\times\frac{2c}{2c}\times\frac{d}{d}=\frac{8b^2cd}{6cd}.$$

$\therefore\ \frac{3ad}{6cd}, \frac{18bc}{6cd}, \frac{8b^2cd}{6cd}$ are the fractions required.

If the denominators have some of them a common measure, find the least common multiple of all the denominators, and multiply the numerator and denominator of each fraction by such a number as will make its denominator equal to that common multiple.

Ex. Reduce $\frac{ab}{cd}, \frac{a^2}{bd^2}, \frac{4x}{3c^2}$, to a common denominator.

Here $3bc^2d^2$ is the least common multiple, and

$$\frac{ab}{cd}=\frac{ab}{cd}\times\frac{3bcd}{3bcd}=\frac{3ab^2cd}{3bc^2d^2}.$$

$$\frac{a^2}{bd^2}=\frac{a^2}{bd^2}\times\frac{3c^2}{3c^2}=\frac{3a^2c^2}{3bc^2d^2}.$$

$$\frac{4x}{3c^2}=\frac{4x}{3c^2}\times\frac{bd^2}{bd^2}=\frac{4bd^2x}{3bc^2d^2}.$$

$\therefore\ \frac{3ab^2cd}{3bc^2d^2}, \frac{3a^2c^2}{3bc^2d^2}, \frac{4bd^2x}{3bc^2d^2}$ are the fractions required.

Proof. *Lem.* 4.

Case V.

To add fractional quantities.

Rule. Reduce the fractions, if necessary, to a common denominator (by Case IV): add together the numerators of the fraction so found, and under their sum place the common denominator.

Ex. Add together $\frac{a+x}{b}$ and $\frac{c}{2d}$.

$$\frac{a+x}{b}=\frac{a+x}{b}\times\frac{2d}{2d}=\frac{2ad+2dx}{2bd}.$$

$$\frac{c}{2d}=\frac{c}{2d}\times\frac{b}{b}=\frac{bc}{2bd}$$

$\therefore\ \frac{2ad+2dx+bc}{2bd}$ is the sum required.

To add mixed numbers, add the fractions by the above rule, and to their sum affix the sum of the integral parts.

Ex. Add together $3a+\frac{18x}{9}$ and $a+\frac{8x}{5}$.

$$\frac{18x}{9}\times\frac{5}{5}=\frac{90x}{45}\ :\ \frac{8x}{5}\times\frac{9}{9}=\frac{72x}{45}$$

$\therefore\ 4a+\frac{162x}{45}=\frac{180a+162x}{45}$ is the sum required.

Case VI.

To subtract fractional quantities.

Rule. Reduce the fractions, if necessary, to a common denominator (by Case IV), subtract the numerators, and under their difference place the common denominator.

Ex. From $\frac{9x}{2}$ take $\frac{2x+1}{3}$.

$$\frac{9x}{2}=\frac{9x}{2}\times\frac{3}{3}=\frac{27x}{6}$$

$$\frac{2x+1}{3}=\frac{2x+1}{3}\times\frac{2}{2}=\frac{4x+2}{6}$$

$\therefore\ \frac{23x-2}{6}$ is the difference required.

Proof. $\frac{a}{b}\pm\frac{c}{d}=\frac{ad}{bd}\pm\frac{bc}{bd}=\frac{ad\pm bc}{bd}.$

CASE VII.

To multiply fractional quantities.

RULE. Multiply the numerators together for the numerator, and the denominators for the denominator of the product.

Ex. 1. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

PROOF. *Ex.* 1. $\frac{a}{b} \times c = \frac{ac}{b}$ (*Lem.* 1). But we had to multiply not by c, but by $\frac{c}{d}$, a quantity d times less than c: the product $\frac{ac}{b}$ is therefore d times too large: hence the true product is $\frac{ac}{b}$ divided by d, *i. e.* $\frac{ac}{bd}$ (*Lem.* 2).

If any of the numerators contain factors common to any of the denominators, those factors may be struck out.

Ex. 2. Multiply together $\frac{3a^2}{2b}$, $\frac{8bc}{ad}$, $\frac{dx}{9c^2}$.

$$\frac{3a^2}{2b} \times \frac{8bc}{ad} \times \frac{dx}{9c^2} = \frac{3a^2 \times 8bc \times dx}{2b \times ad \times 9c^2} = \frac{a \times 4 \times x}{1 \times 1 \times 3c} = \frac{4ax}{3c}.$$

CASE VIII.

To divide fractional quantities.

RULE. Invert the divisor and proceed as in multiplication.

Ex. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

PROOF. $\frac{a}{b} \div c = \frac{a}{bc}$ (*Lem.* 2). But we had to divide not by c but by $\frac{c}{d}$, a quantity d times less than c: the quotient $\frac{a}{bc}$ is therefore d times too large: hence the true quotient is $\frac{a}{bc}$ multiplied by d, *i. e.* $\frac{ad}{bc}$ (*Lem.* 1).

EQUATIONS.

On the Solution of Equations which contain one or more fractions.

Rule. Clear the equation of fractions, by multiplying both sides by the least common multiple of the denominators, and then apply the rules (page 16).

Ex. Solve the equation $\frac{x}{2}+\frac{x}{3}=13-\frac{x}{4}$.

Multiplying by 12 the least common multiple of 2, 3, 4

we have $6x+4x=156-3x$

transposing, $6x+4x+3x=156$

$\therefore\ 13x=156$: and $x=12$.

Proof. Since equals are multiplied by the same, the products are equal.

N. B.—When there are several fractions, it is better to clear the equation of fractions gradually.

Ex. $\frac{6x+8}{11}-\frac{5x+3}{2}=\frac{27-4x}{3}-\frac{6x+18}{4}$.

Multiplying by 4, $\frac{24x+32}{11}-10x-6=\frac{108-16x}{3}-6x-18$.

transposing, $\frac{24x+32}{11}-4x=\frac{108-16x}{3}-12$.

multiplying by 3, $\frac{72x+96}{11}-12x=108-16x-36$.

transposing, $\frac{72x+96}{11}=72-4x$.

multiplying by 11, $72x+96=792-44x$

transposing, $116x=696$, and $\therefore\ x=6$.

Questions producing equations containing fractions.

Ex. 1. What number is that whose third part exceeds its fourth part by 16?

Let $x=$ the number, then $\frac{x}{3}=$ its third part, and $\frac{x}{4}=$ its fourth part, and since the difference of these parts is 16,

$$\frac{x}{3}-\frac{x}{4}=16$$

clearing equation of fractions, $4x-3x=192$

$\therefore\ x=192$, the number required.

Ex. 2. Divide £128 among three persons, so that the first may have three times as much as the second; and the third, one third as much as the first and second together.

Let $x=$ the share of the second, then $3x=$ share of the first, and $\frac{4x}{3}=$ share of the third, and since £128 = the whole sum,

$$x+3x+\frac{4x}{3}=128$$

clearing equation of fractions, $3x+9x+4x=384$

$\therefore\ 16x=384$

whence $x=24$, and $3x=72$, and $\frac{4x}{3}=32$.

Ex. 3. How many trees are there in an orchard containing $\frac{1}{5}$th pear trees, $\frac{3}{7}$th apple trees, and 26 trees of other kinds?

Let $x=$ the number of trees, then $\frac{x}{5}=$ the number of pear trees, and $\frac{3x}{7}=$ the number of apple trees, and since the pear and apple trees with 26 of other kinds equal all in the orchard,

$$\frac{x}{5}+\frac{3x}{7}+26=x$$

Hence $7x+15x+910=35x$

transposing, $35x-7x-15x=910$

$\therefore\ 13x=910$: and $x=70$, the number of trees.

On Simple Equations involving two unknown quantities.

This class of equations is of the form

$$ax+by=c$$
$$dx+ey=f.$$

RULE 1. Clear each equation (if necessary) of fractions, collect the unknown quantities on the left hand, and the known on the right of each equation, so that they shall be reduced to the above form: then multiply or divide each equation by such a quantity, as will make the coefficients of one of the unknown quantities the same in both: then add or subtract, as the signs of that unknown quantity are different or the same, and there will result an equation involving only one unknown quantity, whose value may thence be found: substitute this value in either of the original equations, and there will result an equation involving only the other unknown quantity, whose value may be found.

Ex. 1. $5x+4y=42$
$9x+3y=63.$

Multiplying the 1st equation by 9, and the 2nd by 5,

$$45x+36y=378$$
$$45x+15y=315$$

by subtraction $21y=63$

$$\therefore\ y=3$$

substituting 3 for y $5x+12=42$
in 1st equation $\therefore\ 5x=30$

and $x=6$

$$\therefore\ x=6 \text{ and } y=3.$$

Ex. 2. $\dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5}$

$$8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}.$$

clearing 1st equation of fractions, $15x-25y+30=4x+2y$

transposing, $11x-27y=-30$ (A)

clearing 2nd equation of fractions, $96-3x+6y=6x+4y$

transposing, $9x-2y=96$ (B)

multiplying (A) by 9 and (B) by 11,

$$99x-243y=-270$$
$$99x-22y=1056$$

by subtraction, $-221y=-1326$

$\therefore\ y=6$

substituting 6 for y in (B), $9x-12=96$

$\therefore\ 9x=108$

and $x=12$

$\therefore\ x=12$ and $y=6$.

Rule II. In either of the equations, find the value of one of the unknown quantities in terms of the other and of the known quantities: for it, substitute this value in the other equation, and there will result an equation containing only one unknown quantity, whose value may thence be found: and this value substituted in the expression for the other unknown quantity, will give its value.

Ex. $3x+2y=118$
$x+5y=191$.

By transposing $2y$ in 1st equation and dividing by 3,

$$x=\frac{118-2y}{3} \quad \text{(A)}$$

substituting this value of x in 2nd equation,

$$\frac{118-2y}{3}+5y=191$$

clearing of fraction, $118-2y+15y=573$

transposing, $-2y+15y=573-118$

$\therefore\ 13y=455$ and $y=35$.

substituting 35 for y in (A), $x=\dfrac{118-70}{3}=\dfrac{48}{3}=16$.

$\therefore\ x=16$ and $y=35$.

Rule III. Find from each equation an expression for one of the unknown quantities: put these expressions equal to one another and you will have an equation containing only one unknown quantity, whose value may thence be found: this value substituted in either of the above expressions, will give the value of the other unknown quantity.

Ex. $\dfrac{x+2}{3}+8y=31$

$$\frac{y+5}{4}+10x=192.$$

Clearing 1st equation of fraction, $x+2+24y=93$

transposing, $x=91-24y$ (A)

clearing 2nd equation of fraction, $y+5+40x=768$

transposing, $40x=763-y$

$$\therefore\ x=\frac{763-y}{40}\ \text{(B)}$$

equating the expressions for x, $91-24y=\dfrac{763-y}{40}$

clearing of fraction, $3640-960y=763-y$

transposing, $959y=2877$

$$\therefore\ y=\frac{2877}{959}=3$$

substituting 3 for y in (A), $x=91-72=19$

$\therefore\ x=19$ and $y=3$.

On the solution of simple equations containing three unknown quantities.

RULE. By Rule (page 28) find from two of the equations an equation, involving only two of the unknown quantities, also from the remaining equation and either of the others find, in like manner, an equation involving the same two unknown quantities: from these two equations, the values of the two unknown quantities may be found, and by substituting those values in any of the original equations, the value of the third unknown quantity may be found.

Ex. $x+y+z=29$

$$x+2y+3z=62$$

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10$$

Subtracting 1st equation from 2nd,

$$y+2z=33 \quad \text{(A)}.$$

Multiplying 1st by 6, and 3rd equation by 12,

$$6x+6y+6z=174$$

$$6x+4y+3z=120$$

by subtraction, $2y+3z=54$

multiplying (A) by 2, $2y+4z=66$

by subtraction, $z=12$

substituting for z in (A), $y+24=33$

$$\therefore y=9$$

substituting for y and z in 1st equation,

$$x+9+12=29$$

$$\therefore x=8$$

$$\therefore x=8 : y=9 : z=12.$$

Questions producing equations containing two unknown quantities.

Ex. 1. What fraction is that, to the numerator of which if 1 be added, its value will be $\frac{1}{3}$: but if 1 be added to the denominator, its value will be $\frac{1}{4}$?

Let $\frac{x}{y}$ = the fraction.

then $\frac{x+1}{y}$ = the fraction with 1 added to the numerator,

and $\frac{x}{y+1}$ = ... denominator.

Therefore $\frac{x+1}{y}=\frac{1}{3}$ and $\frac{x}{y+1}=\frac{1}{4}$

Hence $3x+3=y$ and $4x=y+1$

transposing $4x-y=1$

and $3x-y=-3$

by subtraction, $x\ =\ 4$

and $y=4x-1=16-1=15$

$\therefore\ \frac{4}{15}$ is the fraction required.

Ex. 2. A garrison consists of 1250 men, partly infantry, and partly cavalry. Each horse-soldier's pay is £5 per month, and each foot-soldier's £3. If the monthly pay of the garrison be £4150, how many horse and how many foot-soldiers does it contain?

Let x = the number of infantry.

and y = the number of cavalry.

then $3x$ = monthly pay of infantry.

and $5y$ = monthly pay of cavalry.

Therefore $x+y=1250$

and $3x+5y=4150$

multiplying 1st equation by 3, $3x+3y=3750$

and $3x+5y=4150$

by subtraction, $2y=\ 400$

$\therefore\ y=\ 200$ cavalry.

and $x=1250-200=1050$ infantry.

INVOLUTION.

Involution is the raising a quantity to any proposed power.

On the involution of simple quantities.

RULE 1. If the sign of the given quantity be $+$, *all* the powers will be $+$: if it be $-$, the *even* powers will be $+$, and the *odd* powers will be $-$.

2. Raise the coefficients as in whole numbers.

3. Multiply the index of each factor of the literal part by the index of the proposed power.

Ex. 1. The 2nd power of $2a^3 = 4a^{3\times2} = 4a^6$.

2. The 3rd power of $-7ab^5 = -343a^{1\times3}b^{5\times3} = -343a^3b^{15}$.

To raise a fraction, raise both the numerator and denominator to the given power by the above rule.

Ex. The 4th power of $\dfrac{2ab^{-2}}{x^3y^n} = \dfrac{16a^{1\times4}b^{-2\times4}}{x^{3\times4}y^{n\times4}} = \dfrac{16a^4b^{-8}}{x^{12}y^{4n}}$.

PROOF. RULE 1. If $+a$ be multiplied by itself *once, twice, three times*, &c., the results will be $+a^2$, $+a^3$, $+a^4$, &c. where the signs of *all* the powers are $+$. If the same be done with $-a$, the 2nd power will be $+a^2$: the 3rd, $-a^3$: the 4th, $+a^4$: the 5th, $-a^5$, &c., *i. e.* the *even* powers are $+$, and the *odd* $-$.

RULE 3. The 2nd power of $a = a\times a = a^2$, *i. e.* $a^{1\times2}$,

$\therefore$ 3rd power of $a^2 = a^2\times a^2\times a^2 = a^6$, *i.e.* $a^{2\times3}$

$\therefore$ nth power of $a^m = a^m\times a^m\times a^m\times$ &c. to n factors $= a^{mn}$ *i.e.* $a^{m\times n}$.

On the involution of compound quantities.

Rule. A compound quantity may be raised by multiplying the quantity by itself as many times save one as is denoted by the index of the power, but more readily by the Binomial Theorem, which will be given hereafter.

Ex. $a+b$ . . The root.

$$\begin{array}{l}
a+b \\
\hline
a^2+ab \\
\quad\ +ab+b^2 \\
\hline
a^2+2ab+b^2 \text{ . . 2nd power.} \\
a+b \\
\hline
a^3+2a^2b+ab^2 \\
\quad\ +a^2b+2ab^2+b^3 \\
\hline
a^3+3a^2b+3ab^2+b^3 \text{ . . 3rd power.} \\
a+b \\
\hline
a^4+3a^3b+3a^2b^2+ab^3 \\
\qquad +a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
a^4+4a^3b+6a^2b^2+4ab^3+b^4 \text{ . . 4th power and so on.}
\end{array}$$

To find the 2nd, 3rd, &c. power of $2a-b+3x+y$, we may divide it into two parts, thus.

$$\begin{array}{l}
(2a-b)+(3x+y) \text{ . . The root.} \\
(2a-b)+(3x+y) \\
\hline
(2a-b)^2+(2a-b)(3x+y) \\
\qquad +(2a-b)(3x+y)+(3x+y)^2 \\
\hline
(2a-b)^2+2(2a-b)(3x+y)+(3x+y)^2 \text{ . . 2nd power}
\end{array}$$

$=4a^2-4ab+b^2+12ax-6bx+4ay-2by+9x^2+6xy+y^2.$

EVOLUTION.

Evolution is the finding the root of any given quantity.

Case I.

On the evolution of simple quantities.

Rule 1. If the sign of the given quantity be +, any *even* root will be + or —; and any *odd* root must be +: if the sign of the given quantity be —, the *odd* roots will be —, but no *even* root can be taken.

2. Extract the root of the coefficients as in whole numbers.

3. Divide the index of each factor of the literal part by the index of the proposed root.

Ex. 1. The square root of $16a^2 = \pm 4a^{\frac{2}{2}} = \pm 4a$.

2. The 3rd root of $-64a^6b^3c = -4a^{\frac{6}{3}} b^{\frac{3}{3}} c^{\frac{1}{3}} = -4a^2bc^{\frac{1}{3}}$.

To extract the root of a fraction, extract the root of both numerator and denominator.

Ex. The 4th root of $\dfrac{a^4b^8}{c^{12}d^m} = \dfrac{a^{\frac{4}{4}}\, b^{\frac{8}{4}}}{c^{\frac{12}{4}}\, d^{\frac{m}{4}}} = \dfrac{ab^2}{c^3 d^{\frac{m}{4}}}$.

Proof. Rule 1. Since $a^2 = +a \times +a$ or $-a \times -a$,

$\therefore \sqrt{a^2} = +a$ or $-a$.

Since $a^3 = +a \times +a \times +a$ only,

$\therefore \sqrt[3]{a^3} = +a$ only.

Since the *odd* powers of $-a$ are $-a^3$, $-a^5$, $-a^7$, &c.

$\therefore \sqrt[3]{-a^3}$, $\sqrt[5]{-a^5}$, $\sqrt[7]{-a^7}$, &c. are each $-a$.

Since all *even powers* are +, there can be no *even root* of a quantity whose sign is —.

3. Since the square of $a = a^2$ $\therefore$ square root of $a^2 = a$ *i. e.* $a^{\frac{2}{2}}$.

.... 10th power of $a^2 = a^{20}$ $\therefore$ 10th root of $a^{20} = a$ *i.e.* $a^{\frac{20}{10}}$.

.... mth power of $a^n = a^{mn}$ $\therefore$ mth root of $a^{mn} = a^n$ *i. e.* $a^{\frac{mn}{m}}$.

Case II.

To extract the square root of compound quantities.

Rule. Arrange the terms according to the powers of some letter: find the square root of the first term and put it in the root.

Place the square of the root thus found under the first term: subtract and bring down the next two terms.

Double the root already found, and divide the first term of the remainder by it, and place the result in the root and also in the divisor.

Multiply the divisor thus found by the new root, and subtract the product from the remainder: then proceed as before, till all the terms are brought down.

Proof. By comparing a quantity as $a+b$ with its square $a^2+2ab+b^2$, and retracing our steps we arrive at this rule. $(a+b)^2=a^2+2ab+b^2$ $\therefore$ $\sqrt{(a^2+2ab+b^2)}=a+b$.

Let us then see how $a+b$ can be obtained from $a^2+2ab+b^2$.

$$
\begin{array}{rl}
 & a^2+2ab+b^2\,(a+b \\
 & a^2 \\
\hline
2a+b) & 2ab+b^2 \\
 & 2ab+b^2 \\
\hline
 & \cdot \quad \cdot
\end{array}
$$

Arrange $a^2+2ab+b^2$ according to the powers of a: the first quantity sought is a, which, we observe, is the square root of the first term a^2: we therefore take the square of the first term and put it in the root, then we square a, place a^2 under the first term, subtract it, and bring down the next two terms $2ab+b^2$. The next quantity sought is $+b$, and we get it by dividing $2ab$ by $2a$, twice the root found: we put $+b$ in the root and divisor, and multiplying by $+b$, and subtracting we have no remainder.

If there be more terms we should have considered $(a+b)$ as the quotient, and proceeded as above.

Ex. Extract the square root of $a^2+2ab+b^2+2ac+2bc+c^2$.

$$
\begin{array}{lll}
 & & a^2+2ab+b^2+2ac+2bc+c^2(a+b+c \\
 & & a^2 \\
 & 2a+b) & 2ab+b^2 \\
 & & 2ab+b^2 \\
2(a+b)+c \text{ or } & 2a+2b+c) & 2ac+2bc+c^2 \\
 & & 2ac+2bc+c^2 \\
 & & \cdot \quad \cdot \quad \cdot
\end{array}
$$

When the root cannot be found exactly, as many terms of it as desired may be found by the above method.

Ex. Extract the square root of $1+x$.

$$
\begin{array}{ll}
 & 1+x\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\text{\&c.}\right. \\
 & 1 \\
2+\frac{x}{2}) & x \\
 & x+\frac{x^2}{4} \\
2+x-\frac{x^2}{8}) & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
2+x-\frac{x^2}{4}+\frac{x^3}{16}) & \frac{x^3}{8}-\frac{x^4}{64} \\
 & \frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{256} \\
 & -\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{256}
\end{array}
$$

CASE III.

To extract the cube root of compound quantities.

RULE. Arrange the terms according to the powers of some letter. Find the cube root of the first term and set it in the root.

Place the cube of the root thus found under the first term: subtract and bring down the next three terms.

Square the root already found and multiply it by 3: divide the first term of the remainder by it and place the result in the root.

Multiply the terms of the root thus found together and their product by 3, square the last term of the root, and place the sum of these results in the divisor.

Multiply the divisor thus found by the last term of the root: subtract the product from the remainder, and so on till there is no remainder.

PROOF. Since $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, the inverse process will be similar to that shewn in square root.

$$
\begin{array}{rl}
 & a^3+3a^2b+3ab^2+b^3(a+b \\
 & a^3 \\
\hline
3a^2+3ab+b^2) & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\
\hline
 & \cdot \quad \cdot \quad \cdot
\end{array}
$$

Ex. Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$$
\begin{array}{rl}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1(x^2-2x+1 \\
 & x^6 \\
\hline
3x^4-6x^3+4x^2) & -6x^5+15x^4-20x^3 \\
 & -6x^5+12x^4-8x^3 \\
\hline
3x^4-12x^3+15x^2-6x+1) & 3x^4-12x^3+15x^2-6x+1 \\
 & 3x^4-12x^3+15x^2-6x+1 \\
\hline
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}
$$

QUADRATIC EQUATIONS.

Pure quadratics are such as contain *only* the square of the unknown quantity: as $5x^2-43=3$.

Adfected quadratics are such as contain both the square and first power of the unknown quantity: as $x^2-6x=7$.

Case I.

To solve Pure quadratic equations.

Rule. Clear the equation of fractions (if necessary): transpose as in simple equations, and extract the root of both sides.

Ex. $5x^2-43=2$

transposing, $5x^2=2+43=45$

$$\therefore x^2=\frac{45}{5}=9 \text{ and } x=\pm 3.$$

N. B.—Any equation containing only one power of the unknown quantity may be solved in this manner.

Ex. $7x^3-35=6x^3-8$

transposing, $7x^3-6x^3=35-8$

$$\therefore x^3=27 \text{ and } x=3.$$

Case II.

To solve Adfected quadratic equations.

Rule 1. Clear the equation of fractions (if necessary): transpose so that the unknown quantities shall be on the left, and the known on the right, arranging the unknown quantities according to their powers, placing the square first.

Add to both sides, the square of half the coefficient of the second term, and the side involving the unknown quantity will be a complete square.

Extract the square root of both sides, and there will result a simple equation from which the value of the unknown quantity may be found.

Ex. $x^2=33-8x$.

Transposing, $x^2+8x=33$

completing the square, $x^2+8x+16=33+16=49$

extracting the roots, $x+4=\pm 7$

$\therefore\ x=\pm 7-4=3$ or -11.

2. If the term involving the square of the unknown quantity have a coefficient, all the equation must be divided by that coefficient.

Ex. $5x^2+3=159+4x$.

Transposing, $5x^2-4x=156$

dividing by 5, $x^2-\frac{4}{5}x=\frac{156}{5}$

completing the square, $x^2-\frac{4}{5}x+\frac{4}{25}=\frac{156}{5}+\frac{4}{25}=\frac{784}{25}$

extracting the roots, $x-\frac{2}{5}=\pm\frac{28}{5}$

$$\therefore x=\pm\frac{28}{5}+\frac{2}{5}=\frac{30}{5}\text{ or }-\frac{26}{5}=6\text{ or}-\frac{26}{5}.$$

3. Equations in which the index of the unknown quantity in one term is double of that in the other may be thus solved.

Ex. $x^6-4x^3=621$.

Completing the square, $x^6-4x^3+4=621+4=625$

extracting the roots, $x^3-2=\pm 25$

$\therefore\ x^3=\pm 25+2=27$ or -23

$\therefore\ x=3$ or $\sqrt[3]{-23}$.

4. With binomial quantities and others, when the index of one is double that of the other, a similar method may be used.

Ex. $(x-5)^2 = 420-(x-5)$.

Transposing, $(x-5)^2 + (x-5) = 420$

Let $x-5=y : (x-5)^2=y^2$

then $y^2+y=420$

completing the square, $y^2+y+\frac{1}{4}=420+\frac{1}{4}=\frac{1681}{4}$

extracting the roots, $y+\frac{1}{2}=\pm\frac{41}{2}$

$\therefore y=\pm\frac{41}{2}-\frac{1}{2}=\frac{40}{2}$ or $-\frac{42}{2}=20$ or -21

for y substituting its value, we have

$x-5=20$ or -21

and $x=25$ or -16.

5. In some cases it is necessary to add or subtract some quantity on both sides to reduce the equation to the above form.

Ex. $x^2-2x+6\sqrt{x^2-2x+5}=11$.

Adding 5, $x^2-2x+5+6\sqrt{x^2-2x+5}=16$

Let $\sqrt{x^2-2x+5}=y \;\therefore\; x^2-2x+5=y^2$

then $y^2+6y=16$

completing the square, $y^2+6y+9=16+9=25$

extracting the roots, $y+3=\pm 5$

and $y=\pm 5-3=2$ or -8

$\therefore \sqrt{x^2-2x+5}=2$ or -8

and $x^2-2x+5=4$ or 64

transposing, $x^2-2x=-1$ or 59

completing the square, $x^2-2x+1=1-1$ or $1+59=0$ or 60

extracting the roots, $x-1=0$ or $\pm\sqrt{60}$

$\therefore x=1$ or $1\pm\sqrt{60}$.

Problems producing Quadratic Equations.

1. What number is that which, when added to its square root, will make 210?

Let $x^2 =$ the number,

then $x =$ the square root of the number.

$$\therefore \quad x^2 + x = 210$$

Completing the square, $x^2 + x + \frac{1}{4} = 210 + \frac{1}{4} = \frac{841}{4}$

extracting the roots, $x + \frac{1}{2} = \pm \frac{29}{2}$

$$\therefore \quad x = \pm \frac{29}{2} - \frac{1}{2} = \frac{28}{2} \text{ or } - \frac{30}{2} = 14 \text{ or } - 15$$

and $x^2 = 196$ or 225, the number required.

2. A buys a number of oxen for 80 guineas: if he had bought 4 more for the same money, he would have bought them a guinea a head cheaper. How many oxen did he buy?

Let $x =$ the number of oxen bought,

then $\frac{80}{x} =$ the price of each ox.

Now $x + 4 =$ the number and 4 more,

and $\frac{80}{x+4} =$ the price of each of this latter number.

But the latter price is less than the former by one guinea,

$$\therefore \quad \frac{80}{x+4} = \frac{80}{x} - 1$$

Clearing fractions, $80x = 80x + 320 - x^2 - 4x$

transposing, $x^2 + 4x = 320$

completing the square, $x^2 + 4x + 4 = 324$

extracting the roots, $x + 2 = 18$

$\therefore \ x = 18 - 2 = 16$, the number of oxen bought.

ON FRACTIONAL INDICES OR RADICAL SIGNS.

1. By definition, page 4:

$a^{\frac{2}{3}}$ or $\sqrt[3]{a^2}$ denotes the 3rd root of a^2.

$a^{\frac{m}{n}}$ or $\sqrt[n]{a^m}$ .... nth...... a^m.

2. Hence we may use a fractional index or radical sign, as may be most convenient.

3. When the root cannot be exactly extracted, the *quantities* are called irrational or surds: as $\sqrt{2}$, $\sqrt[3]{3}$.

Case I.

To represent a quantity with an integral index under the form of one with a proposed root.

Rule. Raise the quantity to the power corresponding to the proposed root, and over it place a fractional index, or before it a radical sign indicating the proposed root.

Ex. Represent $6ax^2$ in the form of the square root.

$$6ax^2 = (36a^2x^4)^{\frac{1}{2}} \text{ or } \sqrt{36a^2x^4}.$$

Proof. $a^1 = a^{\frac{2}{2}}$ or $\sqrt{a^2} = a^{\frac{3}{3}}$ or $\sqrt[3]{a^3} = a^{\frac{m}{m}}$ or $\sqrt[m]{a^m}$.

Cor. Quantities having indices with *different* denominators or having *different* radical signs, may be changed into equivalent ones, with the *same* denominators or radical signs: thus $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$, or $\sqrt[2]{a}$ and $\sqrt[3]{b} = a^{\frac{3}{6}}$ and $b^{\frac{2}{6}}$, or $\sqrt[6]{a^3}$ and $\sqrt[6]{b^2}$.

Case II.

To reduce a quantity with a fractional index or radical sign to its simplest form.

Rule. Resolve the quantity into two factors, one of which is the greatest power corresponding to the root indicated by the denominator: extract the root of this power and place the result before the remaining part.

Ex. Reduce $\sqrt[3]{81a^3b^7x}$ to its simplest form.

$$(81a^3b^7x)^{\frac{1}{3}} = (27a^3b^6 \times 3bx)^{\frac{1}{3}} = 3ab^2(3bx)^{\frac{1}{3}} \text{ or } 3ab^2\sqrt[3]{3bx}.$$

If the quantity be fractional, multiply the numerator and denominator by such quantity as will make the denominator a complete power of the kind required.

Ex. Reduce $2\sqrt{\dfrac{3x}{5a^5}}$ to its simplest form.

$$2\left(\frac{3x}{5a^5}\right)^{\frac{1}{2}}=2\left(\frac{15ax}{25a^6}\right)^{\frac{1}{2}}=2\left(\frac{1}{25a^6}\times 15ax\right)^{\frac{1}{2}}$$

$$=\frac{2}{5a^3}(15ax)^{\frac{1}{2}} \text{ or } \frac{2}{5a^3}\sqrt{15ax}.$$

N. B.—A coefficient (as 2 in this example) must be multiplied into the root of the factor.

Case III.

Addition and Subtraction.

Rule. Reduce the quantities to their simplest forms, add or subtract the rational parts and to the sum or difference annex the irrational parts.

Ex. Add together $2\sqrt{a^3b}$ and $3\,(a^7b^5)^{\frac{1}{2}}$

$2\,(a^3b)^{\frac{1}{2}}=2(a^2\times ab)^{\frac{1}{2}}=2a(ab)^{\frac{1}{2}}$ or $2a\sqrt{ab}$

$3\,(a^7b^5)^{\frac{1}{2}}=3(a^6b^4\times ab)^{\frac{1}{2}}=3a^3b^2(ab)^{\frac{1}{2}}$ or $3a^3b^2\sqrt{ab}$.

$\therefore$ $(2a+3a^3b^2)\,(ab)^{\frac{1}{2}}$ or $(2a+3a^3b^2)\sqrt{ab}$ is the sum required.

Similarly $(2a-3a^3b^2)\,(ab)^{\frac{1}{2}}$ or $(2a-3a^3b^2)\sqrt{ab}$ would be the difference.

N. B.—If the irrational parts be not the same, the quantities can only be joined by the sign $+$ or $-$1.

Ex. From $2\sqrt{a^2b}$ take $3(a^3b)^{\frac{1}{3}}$.

$2\sqrt{a^2b}=2(a^2\times b)^{\frac{1}{2}}=2ab^{\frac{1}{2}}$ or $2a\sqrt{b}$.

$3(a^3b)^{\frac{1}{3}}=3(a^3\times b)^{\frac{1}{3}}=3ab^{\frac{1}{3}}$ or $3a\sqrt[3]{b}$.

$\therefore$ $2a\sqrt{b}-3a\sqrt[3]{b}$ is the difference required.

Case IV.

Multiplication.

Rule. The same as for quantities with integral indices, Pp. 8, 9, 10.

Ex. 1. Multiply $-a^{\frac{1}{2}}b^{\frac{5}{2}}$ by $3a^{-\frac{1}{3}}b^{\frac{7}{3}}$

$$-a^{\frac{1}{2}}b^{\frac{5}{2}}\times 3a^{-\frac{1}{3}}b^{\frac{7}{3}}=-3a^{\frac{1}{2}-\frac{1}{3}}b^{\frac{5}{2}+\frac{7}{3}}=-3a^{\frac{1}{6}}b^{\frac{29}{6}}.$$

Ex. 2. Multiply $x-\sqrt{xy}+y$ by $\sqrt{x}+\sqrt{y}$

$$\begin{array}{l}
x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y \\
x^{\frac{1}{2}}+y^{\frac{1}{2}} \\
\hline
x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y \\
\qquad +xy^{\frac{1}{2}}-x^{\frac{1}{2}}y+y^{\frac{3}{2}} \\
\hline
x^{\frac{3}{2}} \quad * \quad\quad * \;+y^{\frac{3}{2}} \\
\hline
\end{array}$$

$\therefore$ $x^{\frac{3}{2}}+y^{\frac{3}{2}}$ *or* $\sqrt{x^3}+\sqrt{y^3}$, the product required.

Proof. $a^{\frac{p}{q}}\times a^{\frac{r}{s}}=a^{\frac{p}{q}+\frac{r}{s}}$.

For let $m=\dfrac{p}{q}$ and $n=\dfrac{r}{s}$

then $mqs=ps$ and $nqs=qr$

$\therefore$ $a^{mqs}\times a^{nqs}=a^{ps}\times a^{qr}=a^{ps+qr}$ (Page 8) and extracting the qsth root of both sides,

$$a^m\times a^n \text{ i.e. } a^{\frac{p}{q}}\times a^{\frac{r}{s}}=a^{\frac{ps+qr}{qs}}=a^{\frac{p}{q}+\frac{r}{s}}.$$

Case V.

Division.

Rule. Same as in Pp. 11 and 13.

Ex. 1. Divide $\sqrt{ax^3}$ by $(bx)^{\frac{1}{2}}$.

$$a^{\frac{1}{2}}x^{\frac{3}{2}}\div b^{\frac{1}{2}}x^{\frac{1}{2}}=\frac{a^{\frac{1}{2}}x^{\frac{3}{2}-\frac{1}{2}}}{b^{\frac{1}{2}}}=\frac{a^{\frac{1}{2}}x}{b^{\frac{1}{2}}} \text{ or } x\sqrt{\frac{a}{b}}.$$

Ex. 2. Divide $a\sqrt{x}-\sqrt{bx}+ay^{\frac{1}{2}}-\sqrt{by}$ by $\sqrt{x}+\sqrt{y}$.

$$x^{\frac{1}{2}}+y^{\frac{1}{2}})ax^{\frac{1}{2}}-b^{\frac{1}{2}}x^{\frac{1}{2}}+ay^{\frac{1}{2}}-b^{\frac{1}{2}}y^{\frac{1}{2}}(a-b^{\frac{1}{2}}$$

$$\begin{array}{cc} ax^{\frac{1}{2}} & +ay^{\frac{1}{2}} \\ \hline -b^{\frac{1}{2}}x^{\frac{1}{2}} & -b^{\frac{1}{2}}y^{\frac{1}{2}} \\ -b^{\frac{1}{2}}x^{\frac{1}{2}} & -b^{\frac{1}{2}}y^{\frac{1}{2}} \\ \hline * & * \\ \hline \end{array}$$

$\therefore$ $a-b^{\frac{1}{2}}$ or $a-\sqrt{b}$ is the quotient.

Proof. Since $a^{\frac{p}{q}+\frac{r}{s}}=a^{\frac{p}{q}}\times a^{\frac{r}{s}}$ (Case iv.)

$$a^{\frac{p}{q}+\frac{r}{s}}\div a^{\frac{p}{q}}=a^{\frac{r}{s}}=a^{\left(\frac{p}{q}+\frac{r}{s}\right)-\frac{p}{q}}.$$

Case VI.

Involution.

Rule. Same as in page 33.

Ex. 1 Raise $\sqrt[4]{ax^3}$ to the 2nd power.

The 2nd power of $(ax^3)^{\frac{1}{4}}=(ax^3)^{\frac{1}{4}\times 2}=(ax^3)^{\frac{1}{2}}$ or $\sqrt{ax^3}$ or $a^{\frac{1}{2}}x^{\frac{3}{2}}$

Ex. 2. Raise $\frac{a}{b^2}\sqrt[5]{(c+x)^3}$ to the 4th power.

The 4th power of $\frac{a}{b^2}(c+x)^{\frac{3}{5}}=\frac{a^4}{b^8}(c+x)^{\frac{12}{5}}$ or $\frac{a^4}{b^8}\sqrt[5]{(c+x)^{12}}$

Proof. $\left.a^{\frac{p}{q}}\right|^{r}=a^{\frac{p}{q}}\times a^{\frac{p}{q}}\times$ &c. r times$=a^{\frac{p}{q}+\frac{p}{q}+\text{\&c. } r \text{ times}}$

$=a^{\frac{pr}{q}}$.

CASE VII.

Evolution.

RULE. Same as in page 35.

Ex. Extract the square root of $81a^{\frac{3}{4}}\sqrt[3]{b^2}$.

The square root of $81a^{\frac{3}{4}}b^{\frac{2}{3}} = \pm 9a^{\frac{3}{4}\times\frac{1}{2}}b^{\frac{2}{3}\times\frac{1}{2}} = \pm 9a^{\frac{3}{8}}b^{\frac{1}{3}}$.

PROOF. $a^{\frac{pr}{q}}$ = the rth power of $a^{\frac{p}{q}}$ $\therefore$ rth root of $a^{\frac{pr}{q}}$ $= a^{\frac{p}{q}} = a^{\frac{pr}{q}\times\frac{1}{r}}$.

Equations involving quantities with fractional indices or radical signs.

1. An equation, involving a quantity with fractional index or radical sign, may often be reduced to a simple equation by transposing so as to get that quantity on one side by itself, and then raising both sides to the power denoted by the index or radical sign.

Ex. 1. $\sqrt{\dfrac{2x}{3}} + 5 = 7.$

Transposing, $\sqrt{\dfrac{2x}{3}} = 2$

squaring both sides, $\dfrac{2x}{3} = 4$

$\therefore\ 2x = 12$

and $x = 6.$

Ex 2. $x = \sqrt{2ax + x^2} + a.$

Transposing, $x - a = \sqrt{2ax + x^2}$

Squaring both sides, $x^2 - 2ax + a^2 = 2ax + x^2$

$\therefore\ 4ax = a^2$

$\therefore\ x = \dfrac{a^2}{4a} = \dfrac{a}{4}.$

2. If there be two quantities with fractional indices or radical signs, transpose so as to get the most complex on one side by itself, and proceed as above.

Ex. $\sqrt{x-16}+\sqrt{x}=8.$

Tranposing, $\sqrt{x-16}=8-\sqrt{x}$

Squaring both sides, $x-16=64-16\sqrt{x}+x$

$\therefore\ 16\sqrt{x}=80$

$\therefore\ \sqrt{x}=5$

and $x=25$

3. If the fractional index of one term of the unknown quantity be half that of the other, the equation is a quadratic.

Ex. $3x^{\frac{5}{3}}+x^{\frac{5}{6}}=3104.$

$$x^{\frac{5}{3}}+\frac{x^{\frac{5}{6}}}{3}=\frac{3104}{3}$$

Squaring both sides, $x^{\frac{5}{3}}+\frac{x^{\frac{5}{6}}}{3}+\frac{1}{36}=\frac{3104}{3}+\frac{1}{36}=\frac{37249}{36}$

extracting the squares, $x^{\frac{5}{6}}+\frac{1}{6}=\pm\frac{193}{6}$

$$\therefore\ x^{\frac{5}{6}}=\pm\frac{193}{6}-\frac{1}{6}=32\text{ or }-\frac{97}{3}$$

$$\therefore\ x=(32)^{\frac{6}{5}}\text{ or }\left(-\frac{97}{3}\right)^{\frac{6}{5}}$$

ARITHMETICAL PROGRESSION.

1. An Arithmetical Progression is a series of quantities which increase or decrease successively by the same quantity, as 3, 5, 7, 9, 11, &c., which increase by 2, and 1, —2, —5, —8, &c., which decrease by —3.

2. The quantity by which the series increases or decreases is called the *common difference*, and may be found by subtracting the former of any two consecutive terms from the latter.

3. The first term and the common difference being given, the series is known. Thus, if the first term be 2, and the common difference 3, the series will be 2, 5, 8, 11, &c. Also if the first term be a, and the common difference b, the series will be

$$a, a+b, a+2b, a+3b, a+4b, a+5b, \&c.$$

4. Here, we observe that a stands first in each term, and that the coefficient of b is always one less than the number of terms, being 1 in the 2nd term, 2 in the 3rd, 3 in the 4th, &c.

5. Any term may be found by adding to the first term a, the common difference multiplied by a number less by one than the number of the terms: thus the

16th term would be $a+15b$.

100th .. .. .. $a+99b$.

nth .. .. .. $a+\overline{n-1}.b$.

6. If the first term $=a$: common difference $=b$: the number of terms $=n$: and the last or nth term be called l, then $l=a+\overline{n-1}.b$

7. Any three of the four quantities a, b, n, l, being given, the fourth may be found.

Ex. 1. The first term of an arithmetical series is 1, the number of terms 5, the last term -11: find the common difference.

$$\text{Generally, } l=a+\overline{n-1}.b.$$
$$\text{Here } a=1:\ n=5:\ l=-11$$
$$\therefore\ -11=1+4b$$
$$\therefore\ 4b=-12$$
$$\text{and } b=-3 \text{ the common difference required.}$$

Ex. 2. The first term of an arithmetical series is 4, the common difference 12, the number of terms 9: find the last term.

$$\text{Generally, } l=a+\overline{n-1}.b.$$
$$\text{Here } a=4:\ b=12:\ n=9$$
$$\therefore\ l=4+8\times 12=4+96=100 \text{ the last term.}$$

To find the sum of an Arithmetical series.

1. Let us first take a particular case, in which the first term is 3, the common difference 2, and the number of terms 8; the series will be 3, 5, 7, 9, 11, 13, 15, 17.

Let $S=$ the sum of the series,

then $S=3+5+7+9+11+13+15+17$, and inverting the terms, $S=17+15+13+11+9+7+5+3$

adding, $2S=20+20+20+20+20+20+20+20$

$=20$ repeated 8 times

$=8\times 20$

$$\therefore\ S=\frac{8\times 20}{2}=80, \text{ the sum required.}$$

In like manner, with the series in which the first term is a, common difference b, number of terms n, and the sum S, we have

$$S = a \quad + (a+b) \quad + (a+2b) + \text{ \&c. } + (a+\overline{n-2}.b) + (a+\overline{n-1}.b)$$

$$S = (a+\overline{n-1}.b) + (a+\overline{n-2}.b) + (a+\overline{n-3}.b) + \text{ \&c. } + (a+b) \quad + a$$

$$2S = (2a+\overline{n-1}.b) + (2a+\overline{n-1}.b) + (2a+\overline{n-1}.b) + \text{\&c. } + (2a+\overline{n-1}.b) + (2a+\overline{n-1}.b)$$

$$= (2a+\overline{n-1}.b) \text{ repeated } n \text{ times } = (2a+\overline{n-1}.b)n$$

$$\therefore\ S = (2a+\overline{n-1}.b)\,\frac{n}{2}.$$

2. Any three of the four quantities a, b, n, S, being given, the fourth may be found.

Ex. 1. Find the sum of the series 1, 4, 7, 10, &c. to 50 terms.

Generally $S = (2a+\overline{n-1}.b)\,\dfrac{n}{2}$.

Here $a=1$: $b=3$: $n=50$

$$\therefore\ S = (2+49\times 3)\,\frac{50}{2} = (2+147)\,25$$

$= 149\times 25 = 3725$, the sum required.

Ex. 2. The sum of an arithmetical series is 55, the first term 5, the common difference 3, find the number of terms.

$$\text{Generally } S=(2a+\overline{n-1}\,.\,b)\frac{n}{2}.$$

$$\text{Here } S=55 : a=5 : b=3$$

$$\therefore\ 55=(10+\overline{n-1}\,.\,3)\frac{n}{2}$$

$$110=(3n+7)n$$

$$3n^2+7n=110$$

$$n^2+\frac{7}{3}n=\frac{110}{3}$$

completing the square, $$n^2+\frac{7}{3}n+\frac{49}{36}=\frac{110}{3}+\frac{49}{36}=\frac{1369}{36}$$

extracting the roots, $$n+\frac{7}{6}=\frac{37}{6}$$

$$n=\frac{37-7}{6}=\frac{30}{6}=5, \text{ the number of terms.}$$

3. Since $l=a+\overline{n-1}.b$ and $S=(2a+\overline{n-1}.b)\dfrac{2}{n}$

$$\therefore\ S=(a+l)\frac{n}{2}.$$

4. Any three of the quantities a, l, n, S, being given, the fourth may be found.

Ex. The first term of an arithmetical series is 3, the sum 80, the number of terms 8, find the last term.

$$\text{Generally, } S=(a+l)\frac{n}{2}.$$

$$\text{Here } S=80 : a=3 : n=8 :$$

$$\therefore\ 80=(3+l)4$$

$$\therefore\ 20=3+l$$

$$\therefore\ l=17, \text{ the last term required.}$$

5. *To insert one arithmetical mean between A and B.*

Let $x =$ common difference,

then A, $A+x$, $A+2x$ will be the series, and

B (the 3rd term) $= A + 2x$

$$\therefore \text{ the mean, } A+x = \frac{2A+2x}{2} = \frac{A+(A+2x)}{2} = \frac{A+B}{2}.$$

6 *To insert* m *arithmetical means between A and B.*

The means being the terms intermediate between the first and the last, the series must have $m+2$ terms, and if $x =$ common difference, the series will be

$$A, \quad A+x, \quad A+2x, \quad \&c. \quad A+mx, \quad A+\overline{m+1}.x.$$

$$\therefore B \text{ (the last term) } = A + \overline{m+1}.x$$

$$\therefore x = \frac{B-A}{m+1},$$

whence the series is known.

Ex. Insert three arithmetical means between 117, and 477.

Let $x =$ common difference,

then 117, $117+x$, $117+2x$, $117+3x$, $117+4x$ is the series.

$$\therefore 477 = 117+4x$$

$$\therefore \quad x = \frac{477-117}{4} = \frac{360}{4} = 90$$

and 117, 207, 297, 387, 477, is the series required.

GEOMETRICAL PROGRESSION.

1. A Geometrical Progression is a series of quantities, which increase or decrease by a common multiplier or divisor, as 1, 2, 4, 8, 16, &c. whose terms are multiplied by 2,

125, 25, 5, 1, $\frac{1}{5}$, &c.............. divided by 5.

2. The quantity by which the terms increase or decrease is called the *common ratio*, and may be found by dividing any term by the term immediately before it.

3. If the first term, and the common ratio be given, the series is known. Thus if the first term be 2, and the common ratio be 3, the series will be 2, 6, 18, 54, &c. Also, if the first term be a, and the common ratio be r, the series will be a, ar, ar^2, ar^3, ar^4, &c.

4. Here we observe the first factor in each term is a, and that the index of r is always *one* less than the number of terms: being 1 in the 2nd, 2 in the 3rd, 3 in the 4th, &c.

5. Any term may be found by multiplying the first term by the common ratio raised to a power less by one than the number of terms: thus the

16th term would be ar^{15}.

100th .... ar^{99}.

nth .... ar^{n-1}.

6. Hence, if the first term be a, the common ratio r, the number of terms n, and the last term l, then $l=ar^{n-1}$.

7. Any three of the four quantities a, r, n, l, being given, the fourth may be found.

Ex. The first term of a geometrical series is 1, the common ratio 2, the number of terms 7; find the last term.

$$\text{Generally } l = ar^{n-1}.$$

$$\text{Here } a=1 : r=2 : n=7 :$$

$$\therefore\ l=1\times 2^{7-1}=2^6=64, \text{ the last term required.}$$

To find the sum of a Geometrical series.

1. Let us first take a particular case, in which the first term is 1, the common ratio 2, and the number of terms 8, the series will be 1, 2, 4, 8, 16, 32, 64, 128.

Let $S=$ the sum of the series,

then $S=1+2+4+8+16+32+64+128$, (and $\times$ing by 2) $2S=\quad 2+4+8+16+32+64+128+256.$

Subtracting, $S=256-1.$

$=255$, the sum required.

In like manner, with the series in which the first term is a, common ratio r, number of terms n, and the sum S, we have

$$S=a+ar+ar^2+\text{ \&c. }\quad +ar^{n-2}+ar^{n-1}$$

$$rS=\quad ar+ar^2+\text{ \&c. }\quad +ar^{n-2}+ar^{n-1}+ar^n$$

$$\therefore\ \overline{r-1}.S=ar^n-a$$

$$\text{and } S=\frac{ar^n-a}{r-1}.$$

2. Any three of the four quantities a, r, n, S being given the fourth may be found.

Ex. Find the sum of the series 3, 6, 12, &c. to 7 terms.

$$\text{Generally } S = \frac{ar^n - a}{r - 1}.$$

$$\text{Here } a = 3 : \ r = 2 : \ n = 7$$

$$\therefore \ S = \frac{3 \times 2^7 - 3}{2 - 1} = 3 \times 128 - 3 = 381, \text{ the sum required.}$$

3. Since $l = ar^{n-1}$ and $S = \dfrac{ar^n - a}{r - 1}$

$$\therefore \ S = \frac{rl - a}{r - 1}.$$

4. Any three of the four quantities a, r, l, S being given, the fourth may be found.

Ex. The first term of a geometric series is 1, the common ratio 2, the sum 127, find the last term.

$$\text{Generally } S = \frac{rl - a}{r - 1}$$

$$\text{Here } a = 1 : \ r = 2 : \ S = 127$$

$$\therefore \ 127 = \frac{2l - 1}{2 - 1} = 2l - 1$$

$$\therefore \ l = \frac{128}{2} = 64, \text{ the last term required.}$$

5. *To insert one Geometrical mean between A and B.*

Let $x =$ common ratio,

then A, Ax, Ax^2, will be the series, and

B (the 3rd term,) $= Ax^2$

$\therefore$ the mean, $Ax = \sqrt{A^2x^2} = \sqrt{A \times Ax^2} = \sqrt{AB}$.

6. *To insert* m *Geometrical means between A and B.*

The series will have $m+2$ terms, and if $x =$ common ratio, the series will be

A, Ax, Ax^2, &c. Ax^m, Ax^{m+1}.

$\therefore B$ (the last term) $= Ax^{m+1}$

and $x = \left(\frac{B}{A}\right)^{\frac{1}{m+1}}$

whence the series is known.

Ex. Insert three Geometrical means between 39 and 3159.

Let $x =$ common ratio,

the series will be 39, $39x$, $39x^2$, $39x^3$, $39x^4$.

$\therefore \quad 3159 = 39x^4$

$\therefore \quad x = \left(\frac{3159}{39}\right)^{\frac{1}{4}} = (81)^{\frac{1}{4}} = 3,$

and 39, 117, 351, 1053, 3159 is the series required.

7. *To find the sum of a decreasing Geometrical series continued ad infinitum.*

$$\text{Let } S = a + ar + ar^2 + \text{ \&c., ad inf.}$$
$$\text{then } rS = \quad ar + ar^2 + \text{ \&c., ad inf.}$$
$$\therefore \quad \overline{1-r}\, S = a$$
$$\text{and } S = \frac{a}{1-r}.$$

Ex. Find the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c. to infinity.

$$\text{Generally, } S = \frac{a}{1-r}$$
$$\text{Here } a = 1 : r = \frac{1}{2}$$
$$\therefore \quad S = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2, \text{ the sum required.}$$

8. *To find the value of a Circulating Decimal.*

Rule. Put the decimal in the form of a decreasing geometrical series and sum the series by the above Rule.

Ex. Find the value of .2222 &c.

$$.2222 \text{ \&c.} = \frac{2}{10} + \frac{2}{10^2} + \frac{2}{10^3} + \text{ \&c., ad inf.}$$
$$\text{Generally, } S = \frac{a}{1-r}$$
$$\text{Here } a = \frac{1}{10} : r = \frac{1}{10}$$
$$\therefore \quad 2222 \text{ \&c.} = \frac{\frac{2}{10}}{1-\frac{1}{10}} = \frac{\frac{2}{10}}{\frac{9}{10}} = \frac{2}{9}.$$

4. If the decimal do not begin with the repeating figure, find the sum of the circulating part, and to it add the preceeding part.

Ex. Find the value of .281717 &c.

$$.001717 \text{ \&c.} = \frac{17}{10^4} + \frac{17}{10^6} + \frac{17}{10^8} + \text{\&c. to infinity.}$$

$$\text{Generally, } S = \frac{a}{1-r}$$

$$\text{Here, } a = \frac{17}{10^4} : r = \frac{1}{10^2}$$

$$\therefore .001717 \text{ \&c.} = \frac{\frac{17}{10^4}}{1-\frac{1}{10^2}} = \frac{\frac{17}{10^4}}{\frac{99}{10^2}} = \frac{17}{9900}$$

$$\therefore .281717 \text{ \&c.} = \frac{28}{100} + \frac{17}{9900} = \frac{2789}{9900}.$$

5. The following is a shorter method. Make the decimal $= S$: multiply both sides by the same power of 10 as there are circulating figures, and subtract the upper line from the lower.

Ex. 1. Find the value of .232323 &c.

$$\begin{aligned} \text{Let } S &= .232323 \text{ \&c.} \\ \text{then } 100\,S &= 23.232323 \text{ \&c.} \\ \therefore 99\,S &= 23 \\ \text{and } S &= \frac{23}{99}. \end{aligned}$$

Ex. 2. Find the value of .281717 &c.

$$\begin{aligned} \text{Let } S &= .281717 \text{ \&c.} \\ \text{then } 100\,S &= 28.171717 \text{ \&c.} \\ \therefore 99\,S &= 27.89 \\ \text{and } S &= \frac{27.89}{99} = \frac{2789}{9900}. \end{aligned}$$

BINOMIAL THEOREM.

1. By the Binomial Theorem we can express in a series any power or root of a quantity consisting of two terms.

2. This Theorem (for proof of which see Appendix), is as follows:

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n.\overline{n-1}}{2}x^{n-2}y^2 + \frac{n.\overline{n-1}.\overline{n-2}}{2.3}x^{n-3}y^3$$

$$+\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4}x^{n-4}y^4 + \&c.$$

3. In using this formula, write out, as in the examples below, the values of n, $n-1$, $n-2$, $n-3$, &c. which you will then have ready for use for the coefficients and indices: you will also see, by noticing the number of negative signs (if any) that enter into any coefficient, whether it is $+ve$ or $-ve$.

Ex. 1. Expand $(x+y)^3$.

Here $n=3$

$\therefore$ n, $n-1$, $n-2$, are equal to

3, 2, 1,

and substituting these values in the Theorem,

$$(x+y)^3 = x^3 + 3x^2y + \frac{3.2}{2}xy^2 + \frac{3.2.1}{2.3}y^3.$$

$$= x^3 + 3x^2y + 3xy^2 + y^3.$$

Ex. 2. Expand $(1-x^2)^{\frac{1}{2}}$.

Here $x=1$: $y=-x^2$: $n=\frac{1}{2}$

$\therefore$ n, $n-1$, $n-2$, $n-3$, &c. are equal to

$\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{3}{2}$, $-\frac{5}{2}$, &c.,

and substituting these values in the Theorem,

$$(1-x^2)^{\frac{1}{2}}=1+\frac{1}{2}(-x^2)-\frac{\frac{1}{2}\cdot\frac{1}{2}}{2}(-x^2)^2+\frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{2.3}(-x^2)^3-\&c.$$

$$=1-\frac{1}{2}x^2-\frac{1}{2^3}x^4-\frac{1}{2^4}x^6-\&c.$$

Ex. 3. Expand $(a^2+b^3)^{-\frac{1}{2}}$.

Here $x=a^2$: $y=b^3$: $n=-\frac{1}{2}$

$\therefore$ n, $n-1$, $n-2$, $n-3$, $n-4$, &c. are equal to

$-\frac{1}{2}$, $-\frac{3}{2}$, $-\frac{5}{2}$, $-\frac{7}{2}$, $-\frac{9}{2}$, &c.,

and substituting these values in the Theorem,

$$(a^2+b^3)^{-\frac{1}{2}}=(a^2)^{-\frac{1}{2}}-\frac{1}{2}(a^2)^{-\frac{3}{2}}b^3+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1.2}(a^2)^{-\frac{5}{2}}b^6$$

$$-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{2.3}(a^2)^{-\frac{7}{2}}b^9+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{1.2.3.4}(a^2)^{-\frac{9}{2}}b^{12}-\&c.$$

$$=a^{-1}-\frac{1}{2}a^{-3}b^3+\frac{1.3}{1.2.2^2}a^{-5}b^6-\frac{1.3.5}{1.2.3.2^3}a^{-7}b^9$$

$$+\frac{1.3.5.7}{1.2.3.4.2^4}a^{-9}b^{12}-\&c.$$

4. A quantity consisting of more than two terms may be expanded by considering all the terms but the first as one term.

Ex. Expand $(a+b+c)^4$.

Let $x=a : y=b+c : n=4$

$\therefore$ n, $n-1$, $n-2$, $n-3$ are equal to
4, 3, 2, 1,

and substituting these values in the Theorem,

$$(a+b+c)^4=a^4+4a^3(b+c)+\frac{4.3}{2}a^2(b+c)^2+\frac{4.3.2.}{2.3}a(b+c)^3$$

$$+\frac{4.3.2.1}{2.3.4}(b+c)^4.$$

$$=a^4+4a^3(b+c)+6a^2(b+c)^2+4a(b+c)^3+(b+c)^4.$$

5. *To find any given term in the expansion of a binomial.*

In the expansion of $(x+y)^n$, the

1st term is x^n.

2nd...... $nx^{n-1}y$.

3rd...... $n.\frac{\overline{n-1}}{2}x^{n-2}y^2$.

4th...... $n.\frac{\overline{n-1}}{2}.\frac{\overline{n-2}}{3}x^{n-3}y^3$. &c.

Here we observe that the last term of the numerator of the coefficient is n minus a number less by 2 than the number of terms: being n or $n-0$ in the 2nd term: $n-1$ in the 3rd term: $n-2$ in the 4th term and so on: in like manner the last term of the denominator is 1 less than the number of terms: the index of x is n minus a number less by one than the number of terms: and the index of y is one less than the number of terms: hence the

$$m\text{th term}=\frac{n.\overline{n-1}.\overline{n-2}\ldots\ldots\overline{n-m-2}}{1.\ 2.\ 3\ldots\overline{m-1}.}x^{n-m+1}y^{m-1}.$$

Ex. Find the 5th term in the expansion of $(a-z^2)^{\frac{3}{2}}$.

Here $x=a : y= -z^2 : n=\frac{3}{2} : m=5$

$$\therefore \text{5th term} = \frac{\frac{3}{2}\cdot\left(\frac{3}{2}-1\right)\cdot\left(\frac{3}{2}-2\right)\cdot\left(\frac{3}{2}-3\right)}{1.2.3.4} a^{\frac{3}{2}-4}(-z^2)^{5-1}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}}{1.2.3.4} a^{-\frac{5}{2}}(-z^2)^4$$

$$= \frac{3}{128}a^{-\frac{5}{2}}z^8.$$

PERMUTATIONS.

1. Permutations are the different orders in which any number of things can be placed, when a certain number of them are taken together: thus of *a*, *b*, *c* taken 3 and 3 together, the permutations are *abc*, *acb*, *bac*, *bca*, *cab*, *cba*.

2. *If* R = *the number of permutations of* n − 1 *things taken all together,* nR *will* = *the number of permutations of* n *things taken all together.*

Take first a particular instance. The number of permutations of three things *a*, *b*, *c*, taken all together is 6, as shown above: take an additional quantity *d*: *d* may be placed *either before or after each* of the three quantities in each permutation: thus in the first, we should have *dabc*, *adbc*, *abdc*, *abcd*, and so in the 2nd: hence from each permutation we should have 4 permutations, or altogether 4 × 6 permutations.

Similarly if the number of permutations of $\overline{n-1}$ things taken all together = *R*, as the additional quantity may be placed *either before or after each* of the $n-1$ quantities in each permutation, we shall have *n* different places in each of the *R* permutations, that is, there will be *n* permutations formed out of each of the *R* permutations : hence there will *nR* permutations of *n* things taken all together.

3. *To find the number of permutations of* n *things taken all together.*

Let Q=number of permutations of n things taken together.

R= . . . . $n-1$ . . . .

S= . . . . $n-2$ . . . .

T= . . . . $n-3$ . . . .

&c.= . . . . &c. . . . .

Then (Art. 2) $Q=nR$: $R=\overline{n-1}.S$: $S=\overline{n-2}.T$ &c. till we come to $(n-\overline{n-1})$ or 1 thing, which admits of only one permutation.

Hence $Q=nR=n.\overline{n-1}.S=n.\overline{n-1}.\overline{n-2}.T=$&c.

$$=n.\overline{n-1}.\overline{n-2}.\overline{n-3}\ldots\ldots\ldots 3.2.1.$$

Ex. How many changes may be rung on 8 bells?

Generally $Q=n.\overline{n-1}\ldots\ldots\ldots 3.2.1$

Here $n=8$

$\therefore$ $Q=8.7.6.5.4.3.2.1=40320$ the number required.

4. *To find the number of permutations of* n *things taken* r *and* r *together.*

Let P =number of permutations of n things taken r together.

Q= . . . . $n-1$ . . . . . . $r-1$. . . . .

R= . . . . $n-2$ . . . . . . $r-2$. . . . .

&c. &c. &c.

W= . . . . $n-\overline{r-1}$ . . . . . . 1 . . . . . .

Now, if we add a new quantity p to $n-1$ things and take r of those together, we shall have Q permutations in which p stands first: we shall also have Q permutations in which each of the n quantities stands first, therefore we shall have nQ permutations of n quantities taken r together.

$$\therefore\ nQ=P$$

Similarly $\overline{n-1}.R=Q$

$\overline{n-2}.S=R$

&c. $=$ &c.

$\therefore$ $P=nQ=n.\overline{n-1}.R=n-\overline{n-1}.\overline{n-2}.S=$ &c.

$$=n.\overline{n-1}.\overline{n-2}\ldots\ldots(n-\overline{r-1})$$

5. *To find the number of permutations of* n *things taken all together, when there* p *of one sort,* q *of another,* r *of another, &c.*

If all the quantities were different, there would be $n \cdot \overline{n-1} \cdot \overline{n-2} \ldots \ldots 3.2.1$ permutations. If the p quantities were different, they would make $(1.2.3 \ldots p)$ permutations: q would make $(1.2.3 \ldots q)$, and r, $(1.2.3 \ldots r)$: but being the same, they each form but one permutation, and $\therefore P$ the whole number of permutations $= \dfrac{n.\overline{n-1}.\overline{n-2} \ldots 3.2.1}{(1.2.3 \ldots p).(1.2.3 \ldots q).(2.3 \ldots r)}$

Ex. What number of permutations may be made of the letters *aabbbcccc*.

Here $n = 9 : p = 2 : q = 3 : r = 4$

$$\therefore P = \frac{9.8.7.6.5.4.3.2.1}{1.2.1.2.3.1.2.3.4} = 1260.$$

COMBINATIONS.

1. Combination is the collection of any number of quantities taken a certain number together, without regard to their order.

2. *To find the number of Combinations of* n *things taken* r *and* r *together.*

Let $N =$ the number. Then, since each combination of r quantities admits of $(1.2.3 \ldots r)$ permutations (by Art. 3, page 64), the whole number of permutations of these n things $= (1.2.3 \ldots r) N$: but the whole number of permutations (by Art. 4, page 64), also equals $(n. \overline{n-1}. \overline{n-2} \ldots \overline{n-r-1})$.

$$\therefore (1.2.3 \ldots r) N = (n. \overline{n-1}. \overline{n-2} \ldots \overline{n-r-1}).$$

$$\text{and } N = \frac{n. \overline{n-1}. \overline{n-2} \ldots \overline{n-r-1}}{1.2.3 \ldots r}.$$

Ex. How many combinations can be made of twenty-six letters taken 3 and 3 together?

Here $n = 26$, and $r = 3$.

$$\therefore N = \frac{26 \times 25 \times 24}{1 \times 2 \times 3} = 2600.$$

LOGARITHMS.

1. In the equation $a^x = y$, in which a is invariable, by varying the value of x, a^x may be made equal to any number whatever.

2. If $a^x = y$, x is called the logarithm of y to base a.

Thus if $a = 10$: and $y = 100, 1000$, &c. successively then since $100 = 10^2$: $1000 = 10^3$, &c. we shall have log. $100 = 2$: log. $1000 = 3$, &c.

3. Since log. $100 = 2$ and log. $1000 = 3$, the logarithm of any number between 100 and 1000 must be greater than 2 and less than 3: thus log. $15 = 1.760913$, $(10)^{1.760913}$ being $= 15$. Similarly, the logarithm of any number between 1000 and 10000, must be greater than 3 and less than 4, and so on.

4. The numbers 10, 100, 1000 &c. are in Geometrical and their logarithms 1, 2, 3, &c. are in Arithmetical Progression. If then we insert geometric means between any two numbers, and the same number of arithmetical means between their logarithms, we shall find the logarithms of those geometric means. By this method we might calculate the logarithm of *all* numbers. Thus log. $10 = 1$: log. $100 = 2$; now the geometric mean between 10 and $100 = 31.622777$, and the arithmetical mean between 1 and 2 is 1.5 $\therefore$ $1.5 =$ log. 31.62777.

5. Log. $185 = 2.2671717$

$$\text{log. } 18.5 = \text{log. } \frac{185}{10} = \text{log. } 185 - \text{log. } 10 = 1.2671717$$

$$\text{log. } 1.85 = \text{log. } \frac{185}{100} = \text{log. } 185 - \text{log. } 100 = .2671717$$

$$\text{log. } .185 = \text{log. } \frac{185}{1000} = \text{log. } 185 - \text{log } 1000 = \bar{1}.2671717$$

$$\text{log. } .0185 = \text{log. } \frac{185}{10000} = \text{log. } 185 - \text{log. } 10000 = \bar{2}.2671717$$

and so on.

Hence the logs. of 185, 18.5, 1.85 &c. have the same decimal part, and differ only in the index or characteristic: this index will be 0, 1, 2, 3, &c. according as the number contains 1, 2, 3, 4 &c. integral figures, and $\bar{1}$, $\bar{2}$ &c. if it be only a *decimal* with 3, 4, &c. places.

6. *The logarithm of a product consisting of any number of factors, is equal to the sum of the logarithms of those factors.*

Let a be the base: x, x', x'' &c. the logarithms of y, y', y'' &c, then we have (by Def. 2).

$$y = a^x : y' = a^{x'} : y'' = a^{x''} \text{ \&c.}$$

$$\therefore \quad yy'y'' \ldots = a^x \times a^{x'} \times a^{x''} \ldots = a^{x+x'+x''\ldots}$$

and log. $yy'y'' \ldots = x + x' + x'' \ldots =$ log. y + log. y' + log. y'' + &c.

7. *The logarithm of a fraction is equal to the logarithm of the numerator—the logarithm of the denominator.*

For, if a be the base, and x and x' the logs of y and y',

$$\text{then } y = a^x \text{ and } y' = a^{x'}$$

$$\therefore \quad \frac{y}{y'} = \frac{a^x}{a^{x'}} = a^{x-x'}$$

$$\therefore \quad \text{log.}\frac{y}{y'} = x - x' = \text{log. } y - \text{log. } y'.$$

8. *The logarithm of any power or root of any quantity is found by multiplying or dividing the logarithm of that quantity by the index of the power or root.*

For, if $y = a^x$ then $y^n = a^{nx}$

$\therefore$ log. $y^n = nx = n$ log. y.

Similarly, $y^{\frac{1}{n}} = a^{\frac{x}{n}}$

$$\therefore \quad \text{log. } y^{\frac{1}{n}} = \frac{1}{n}x = \frac{1}{n} \text{ log. } y.$$

9. Hence by logarithms, the operations of multiplication and division of numbers are reduced to addition and subtraction, and those of involution and evolution to multiplication and division.

Ex. 1. Multiply 37.153 by 4.086.

$$
\begin{aligned}
\text{Log. } 37.153 &= 1.5699939 \\
\text{log. } 4.086 &= 0.6112984 \\
\hline
&\ 2.1812923 \\
&= \text{log. } 151.8072
\end{aligned}
$$

$\therefore$ 151.8072 = product required.

Ex. 2. Find the square of 2.7558.

$$
\begin{aligned}
\text{Log. } 2.7558 &= .4402477 \\
&\qquad\quad 2 \\
\hline
&\ .8804954 \\
&= \text{log. } 7.594434
\end{aligned}
$$

$\therefore$ 7.594434 is the square required.

APPENDIX.

Proof of the Rule for finding the greatest common measure.

LEMMA 1. If one quantity measure two others, it will measure their sum or difference.

Let x measure a by the units in m, and b by the units in n,

then $a = mx$ and $b = nx$

$\therefore$ $a \pm b = (m \pm n)x$, i. e., x measures $a \pm b$ by the units in $m \pm n$.

LEMMA 2. If one quantity measure another it will measure any multiple of it.

Let x measure a by the units in m, them $a = mx$

$\therefore$ $an = mnx$, i. e., x measures an by the units in mn.

To shew how the greatest common measure of a and b may be found, let a be the greater: divide a by b, let p be the quotient with remainder c: divide b the former divisor by c, let q be the quotient with remainder d: then divide c by d, let r be the quotient without remainder: d shall be greatest common measure.

$$\begin{array}{l} b)a(p \\ \underline{bp} \\ \quad c)b(q \\ \quad \underline{cq} \\ \qquad d)c(r \\ \qquad \underline{dr} \\ \qquad \underline{\cdot} \end{array}$$

First, d is a common measure of a and b.

For d measures c, because $c = dr$

$\therefore d$ ...... cq, Lem. 2.

$\therefore d$ ...... $cq+d$ or b, Lem. 1.

$\therefore d$ ...... bp, Lem. 2.

$\therefore d$ ...... $bp+c$ or a, Lem. 1.

$\therefore d$ ...... both a and b.

Secondly, d is the *greatest* common measure of a and b. For, if not, let $d+x$ be the greatest.

Since $d+x$ measures b

$\therefore$ .... bp, Lem. 2.

$\therefore$ .... $a-bp$ or c, Lem. 1.

$\therefore$ .... cq, Lem. 2.

$\therefore$ .... $b-cq$ or d, Lem. 1.

i. e., $d+x$ measures d, which is absurd, since no number greater than d can measure d. Therefore d is the greatest common measure of a and b.

Cor. Whatever number measures a and b, will also measure d.

To find the greatest common measure of three or more quantities a, b, c, &c.

Rule. Find the greatest common measure of a and b: let it be d: then find the greatest common measure of c and d: let that be e: e shall be the greatest common measure of a, b, c, and so on.

For every measure of a and b measures d (by last Cor.)

$\therefore$ .... a, b, c .... d, c

$\therefore$ the greatest measure of d and c is also that of a, b, c, and so on.

To find the least common multiple of two quantities a *and* b.

Rule. Multiply the quantities together, and divide their product by their greatest common measure.

Proof. Let a and b be two quantities whose greatest common measure is x, and let $a = mx : b = nx$. Now the least common multiple of a and b must be that of mx and nx, which is evidently mnx, since m and n have no common divisor: but $mnx = \frac{mnx^2}{x} = \frac{ab}{x} =$ product of a and b, divided by their greatest common measure.

To find the least common multiple of a, b, c.

Let $x =$ greatest common measure of a and b

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{ab}{x}$ and c

then $\frac{ab}{x} =$ least common multiple of a and b

and $\frac{abc}{xy} = \ldots\ldots\ldots\ldots\ldots\ldots \frac{ab}{x}$ and c

and $\therefore \frac{abc}{xy} = \ldots\ldots\ldots\ldots\ldots$ of a, b, c.

Similarly, if the quantities be a, b, c, d, &c.

and $x =$ greatest common measure of a and b

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{ab}{x}$ and c

$z = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{abc}{xy}$ and d.

&c. = &c.

Then $\frac{ab}{x} =$ least common multiple of a and b

$\frac{abc}{xy} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots a, b, c$

$\frac{abcd}{xyz} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots a, b, c, d.$

&c = &c.

SURDS.

1. If two quadratic surds $\sqrt{a}$ and $\sqrt{b}$ cannot be reduced to others which have the same surd, their product is a surd.

For, if possible, let $\sqrt{a} \times \sqrt{b} = ma$

$$\therefore\ ab = m^2a^2$$
$$\therefore\ b = m^2a$$
$$\therefore\ \sqrt{b} = m\sqrt{a},$$

i. e. $\sqrt{b}$ can be reduced so as to have the same surd as $\sqrt{a}$, which is contrary to the hypothesis.

2. One quadratic surd cannot be made up of two others $\sqrt{m}$ and $\sqrt{n}$, which have not the same surd.

For, if possible, let $\sqrt{x} = \sqrt{m} + \sqrt{n}$

$$\therefore\ x = m + n + 2\sqrt{mn}$$

$\therefore\ \dfrac{x - m - n}{2} = \sqrt{mn}$, *i. e.* a rational quantity $=$ a surd, which is impossible.

3. The square root of a quantity cannot be partly rational and partly a surd.

For, if possible, let $\sqrt{a} = b + \sqrt{c}\ \therefore\ a = b^2 + c + 2b\sqrt{c}$.

$\therefore\ \sqrt{c} = \dfrac{a - b^2 - c}{2b}$. *i. e.* a surd quantity $=$ a rational one, which is impossible.

4. If $a \pm \sqrt{x} = b \pm \sqrt{y}$, where a and b are rational and $\sqrt{x}$ and $\sqrt{y}$ are surds, $a = b$ and $\sqrt{x} = \sqrt{y}$: for since $\pm\sqrt{x} = b - a \pm \sqrt{y}$: if $\sqrt{x}$ be not $= \sqrt{y}$, it is partly rational and partly a surd, which is impossible.

5. To extract the square root of $a + \sqrt{b}$.

Let $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$

$\therefore\ a + \sqrt{b} = x + y + 2\sqrt{xy}$

$\therefore\ a = x + y$, and $\sqrt{b} = 2\sqrt{xy}$

$\therefore\ a^2 = x^2 + 2xy + y^2$, and $b = 4xy$

$$\therefore a^2 - b = x^2 - 2xy + y^2$$

and $\sqrt{a^2 - b} = x - y$

but, $\quad a \quad = x + y$

$$\therefore x = \frac{a + \sqrt{a^2 - b}}{2}, \text{ and } y = \frac{a - \sqrt{a^2 - b}}{2}$$

and $\sqrt{a + \sqrt{b}} = \sqrt{\left(\frac{a + \sqrt{a^2 - b}}{2}\right)} + \sqrt{\left(\frac{a - \sqrt{a^2 - b}}{2}\right)}$.

Ex. Find the root of $11 + \sqrt{72}$.

Let $\sqrt{11 + \sqrt{72}} = \sqrt{x} + \sqrt{y}$

$$\therefore 11 + \sqrt{72} = x + 2\sqrt{xy} + y$$

$$\therefore x + y = 11, \text{ and } 2\sqrt{xy} = \sqrt{72}$$

$$\therefore x^2 + 2xy + y^2 = 121$$

$$4xy = 72$$

$$x^2 - 2xy + y^2 = 49$$

$$x - y = 7$$

$$x + y = 11$$

$$\therefore x = 9, \text{ and } y = 2$$

and $\sqrt{11 + \sqrt{72}} = 3 + \sqrt{2}$.

RATIO.

1. Ratio is the relation of one quantity to another of the same kind in respect to magnitude, and is found by dividing one by the other: thus the ratio of a to b (written $a : b$) is expressed by $\frac{a}{b}$.

2. If a be greater than b, the ratio $a : b$ is said to be of *greater inequality*: if less, it is said to be of *lesser inequality*.

3. To compare ratios, reduce the fractions which express them to common denominators and compare the numerators. Thus to compare the ratios of $2 : 3$ and $4 : 5$;

$$\frac{2}{3} = \frac{10}{15} : \frac{4}{5} = \frac{12}{15}$$

$\therefore$ the ratios are to each other as $10 : 12$.

PROPORTION.

4. Proportion is the equality of ratios: thus if the ratio of $a : b$ be equal to that of $c : d$, then a, b, c, d are in proportion: this proportion is expressed by $a : b :: c : d$.

5. If four quantities are proportional, the product of the extremes equals the product of the means: for, if $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$, whence $ad = bc$.

6. If four quantities are proportional, the fourth equals the product of the second and third divided by the first: for if $a : b :: c : d$ then $ad = bc$ and $d = \frac{bc}{d}$. Hence the Rule of Three in Arithmetic.

7. If $a : b :: c : d$

$$\text{then } \because \frac{a}{b} = \frac{c}{d} \text{ and } \because \frac{b}{a} = \frac{d}{c}$$

$$b : a :: d : c \text{ (invertendo)}$$

8. If $a : b :: c : d$

$$\text{then } \because \frac{a}{b} = \frac{c}{d} \text{ and } \because \frac{a}{c} = \frac{b}{d}$$

$$a : c :: b : d \text{ (alternando)}$$

9. If $a : b :: c : d$

$$\text{then } \because \frac{a}{b} = \frac{c}{d} \text{ and } \therefore \frac{a}{b} \pm 1 = \frac{c}{d} \pm 1$$

$$\text{and } \frac{a \pm b}{b} = \frac{c \pm d}{d}$$

$$a + b : b :: c + d : d \text{ (componendo)}$$

$$\text{and } a - b : b :: c - d : d \text{ (dividendo)}$$

10. If $a : b :: c : d$

then $\because \frac{a+b}{b} = \frac{c+d}{d}$ and $\frac{a-b}{b} = \frac{c-d}{d}$

$$\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

and $a+b : a-b :: c+d : c-d$ (componendo and dividendo.)

11. If $a : b :: c : d$ and $e : f :: g : h$

then $\because \frac{a}{b} = \frac{c}{d}$ and $\frac{e}{f} = \frac{g}{h}$ and $\therefore \frac{ae}{bf} = \frac{cg}{dh}$

$$ae : bf :: cg : dh.$$

12. If $a : b :: b : c$

then $\because ac = b^2$ and $\therefore \frac{a}{c} = \frac{a^2}{b^2}$

$a : c :: a^2 : b^2$ or a is to c in the duplicate ratio of a to b.

13. If $a : b :: c : d$.

then $\because \frac{a}{b} = \frac{c}{d}$, and $\therefore \frac{a^m}{b^m} = \frac{c^m}{d^m}$

$$a^m : b^m :: c^m : d^m.$$

Proof of the Binomial Theorem.

LEMMA 1. If $A + Bx + Cx^2 +$ &c. $= a + bx + cx^2 +$ &c. for every possible value of x,

then $A = a$: $B = b$: $C = c$ &c.

For, if the equation be true for *every* value of x, it is true when $x = 0$. Let $x = 0$,

then $A = a$, since Bx, bx, Cx^2, cx^2 &c. each $= 0$

$\therefore Bx + Cx^2 +$ &c. $= bx + cx^2 +$ &c. and dividing by x, $B + Cx +$ &c. $= b + cx +$ &c.

Then, proceeding as before, $B = b$:

In like manner $C = c$, $D = d$ &c.

LEMMA 2. $\frac{a^n - b^n}{a-b}$, when b becomes $= a$, is equal to na^{n-1}: for by division we know that $\frac{a^n-b^n}{a-b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 +$ &c. to n terms: and when $b = a$, this becomes $a^{n-1} + a^{n-1} + a^{n-1} +$ &c. to n terms, or na^{n-1}.

To expand $(x+y)^{\frac{m}{n}}$.

$$(x+y)^{\frac{m}{n}} = x^{\frac{m}{n}} \left(1+\frac{y}{x}\right)^{\frac{m}{n}}$$

$$\text{Let } \frac{y}{x} = a,$$

$$\text{and suppose that } 1 + a = v$$

$$\text{and } 1 + b = z$$

$$\therefore \quad a - b = v - z.$$

Assume $(1+a)^{\frac{m}{n}} = 1 + Aa + Ba^2 + Ca^3 +$ &c. $= v^{\frac{m}{n}}$.

$\therefore \quad (1+b)^{\frac{m}{n}} = 1 + Ab + Bb^2 + Cb^3 +$ &c. $= z^{\frac{m}{n}}$.

$\therefore$ by subtraction, $A(a-b) + B(a^2-b^2) + C(a^3-b^3) +$ &c $= v^{\frac{m}{n}} - z^{\frac{m}{n}}$.

Divide by $a - b = v - z$, and

$$A + B\frac{a^2-b^2}{a-b} + C\frac{a^3-b^3}{a-b} + D\frac{a^4-b^4}{a-b} + \text{\&c.} = \frac{v^{\frac{m}{n}} - z^{\frac{m}{n}}}{v-z}$$

Let $a=b$: and $\therefore v=z$, then by Lem. 2.

$A+2Ba+3Ca^2+4Da^3+$ &c. $=\frac{m}{n}v^{\frac{m}{n}-1}$: multiplying by

$$\begin{array}{l} 1+a \qquad\qquad\qquad = v \\ \hline \left.\begin{array}{r} A+2Ba+3Ca^2+4Da^3+\text{ \&c.} \\ Aa+2Ba^2+3Ca^3+\text{ \&c.} \end{array}\right\} = \frac{m}{n}\, v^{\frac{m}{n}} \\ \hline A+(2B+A)a+(3C+2B)a^2+(4D+3C)a^3+\text{\&c.}=\frac{m}{n}v^{\frac{m}{n}} \end{array}$$

$$=\frac{m}{n}(1+a)^{\frac{m}{n}}=\frac{m}{n}(1+Aa+Ba^2+Ca^3+\text{ \&c.})$$

$$=\frac{m}{n}+\frac{m}{n}Aa+\frac{m}{n}Ba^2+\frac{m}{n}Ca^3+\text{ \&c}$$

$\therefore$ by Lemma 1,

$$A=\frac{m}{n}:$$

$2B+A=\frac{m}{n}A \;\therefore\; 2B=\left(\frac{m}{n}-1\right)A=\frac{m}{n}.\left(\frac{m}{n}-1\right)$, and

$$B=\frac{\frac{m}{n}.\left(\frac{m}{n}-1\right)}{2}$$

$$3C+2B=\frac{m}{n}B \;\therefore\; 3C=\left(\frac{m}{n}-2\right)B=\frac{\frac{m}{n}.\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{2}$$

and $C=\dfrac{\frac{m}{n}.\left(\frac{m}{n}-1\right).\left(\frac{m}{n}-2\right)}{2.3}$. In like manner

$$D=\frac{\frac{m}{n}.\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right).\left(\frac{m}{n}-3\right)}{2.3.4}$$

&c. $=$ &c.

$$\therefore (1+a)^{\frac{m}{n}} = 1 + \frac{m}{n}a + \frac{\frac{m}{n}.\left(\frac{m}{n}-1\right)}{2}a^2 + \frac{\frac{m}{n}.\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{2.3}a^3$$

$$+ \frac{m.\left(\frac{m}{n}-1\right).\left(\frac{m}{n}-2\right)\left(\frac{m}{n}-3\right)}{2.3.4}a^4 + \&c.$$

and substituting $\frac{y}{x}$ for a,

$$(x+y)^{\frac{m}{n}} = x^{\frac{m}{n}}\left(1+\frac{y}{x}\right)^{\frac{m}{n}}$$

$$= x^{\frac{m}{n}}\left(1 + \frac{m}{n}\frac{y}{x} + \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{2}\frac{y^2}{x^2} + \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{2.3}\frac{y^3}{x^3}\right.$$

$$\left. + \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)\left(\frac{m}{n}-3\right)}{2.3.4}\frac{y^4}{x^4}\ \&c.\right)$$

$$= x^{\frac{m}{n}} + \frac{m}{n}x^{\frac{m}{n}-1}y + \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{2}x^{\frac{m}{n}-2}y^2$$

$$+ \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{2.3}x^{\frac{m}{n}-3}y^3$$

$$+ \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)\left(\frac{m}{n}-3\right)}{2.\ 3.\ 4}x^{\frac{m}{n}-4}y^4 + \&c.$$

Cor. 1. If $n=1$

$$(x+y)^m = x^m + m\,x^{m-1}y + \frac{m.(m-1)}{2}x^{m-2}y^2$$

$$+ \frac{m.(m-1)\,(m-2)}{2.3}x^{m-3}y^3 + \frac{m.(m-1).(m-2).(m-3)}{2.3.4}$$

$x^{m-4}y^4 + \&c.$

Cor. 2. If for y we write $-y$, we have

$$(x-y)^m = x^m - mx^{m-1}\,y + \frac{m.(m-1)}{2}\,x^{m-2}\,y^2$$

$$-\ \frac{m.(n-1)\,(n-2)}{2.3}\,x^{m-3}\,y^3 + \&c.$$

Cor. 3. If x and y each $=1$, then $(x+y)^m$ or 2^m

$$= 1 + m + \frac{m.(m-1)}{2} + \frac{m(m-1)(m-2)}{2.3}$$

$$+ \frac{m(m-1)(m-2)(m-3)}{2.3.4} + \&c.$$

i. e. the sum of the coefficients of $(x+y)^m = 2^m$.

Cor. 4. Since (page 62)

$$\text{the } r\text{th term} = \frac{m(m-1)\,(m-2)\ldots(m-\overline{r-2})}{1.\ 2.\ 3 \qquad r-1}\,a^{m-r-1}b^{r-1}$$

$$\text{the } m+1\text{th term} = \frac{m(m-1).(m-2)\ldots(m-\overline{m-2})(m-\overline{m-1})}{1.\ 2.\ 3\ \ldots\ (m-2)(m-1).m}\,a^{m-m}b^m$$

$$= \frac{m(m-1).(m-2)\ldots 2.1}{1.2.3\ldots(m-2).(m-1).m}\,b^m = b^m.$$

In the $(m+2)$th, $(m+3)$th &c. terms, one factor of the numerator will be $m-m$ or 0, and therefore there will be no term after the $(m+1)^m$ of $(x+y)^m$ if m be a positive integer.

Cor. 5. If m be negative or fractional, no factor of any coefficient can equal 0, and therefore the series will continue ad infinitum.

Cor. 6. From comparing the 2nd and mth terms, the 3rd and $(m-1)$th &c. of the expansion of $(x+y)^m$ it will appear that the latter coefficients of the series are the same as those of the corresponding terms taken from the beginning.

FINIS.

[W. Woodward, Printer, Portsea.